Bruno de Finetti and Fuzzy Probability Distributions

Reinhard Viertl
Technische Universität Wien, Austria
r.viertl@tuwien.ac.at

Abstract

Bruno de Finetti stated that probability does not exist in an objective sense. This is the basis for subjective Bayesian inference. For de Finetti probabilities are real numbers from the closed unit interval. Descriptive statistics for fuzzy data yield fuzzy relative frequencies. That is the starting point for modern considerations concerning probability. Recent research results are proposing a general probability concept where probabilities are special fuzzy numbers obeying a generalized form of additivity. This concept of so-called fuzzy probability distributions is explained in the paper.

1 Introduction

In his monumental and basic book *Theory of Probability* Bruno de Finetti gave a deep analysis of probability. One of his main conclusions is that probability is not an objective existing – frequently unknown – quantity, but as he says "probability does not exist, except in the mind". This idea is the basis for all neo-Bayesian statistical methods which were developed in the 20th century.

Another criticism by Bruno de Finetti about probability is concerning countable additivity of probability measures.

These and other comments on the theory of probability raise the question what mathematical model is suitable to describe probability.

2 Current probability models

There are different concepts of probability models. The most popular mathematical model for probability is the concept of probability spaces $(M, \mathcal{E}, \Pr)$, where M is a general set, $\mathcal{E}$ is a sigma field of subsets of M, and Pr a σ-additive and normalized measure on $\mathcal{E}$, i. e.

(1) $\Pr : \mathcal{E} \longrightarrow [0;1]$

(2) $\Pr(M) = 1$

(3) For every countable family $A_1, A_2, \cdots$ of pairwise disjoint events $A_i \in \mathcal{E}$ the following holds

$$\Pr\Big(\bigcup_{n=1}^{\infty} A_n\Big) = \sum_{n=1}^{\infty} \Pr\big(A_n\big).$$

Bruno de Finetti's concept of probability is starting with the events as elementary concept. So he is considering a family $\big(E_i, i \in I\big)$ of so-called events and defines probabilities as real numbers fulfilling finite additivity and the so-called coherence condition. For him all probabilities are conditional on the state of information H, i. e. $\Pr\big(E \mid H\big)$, where new information (for example data) is changing the probability:

(1) $0 \leq \Pr\big(E_i \mid H\big) \leq 1$ for all E_i in the event system

(2) For finitely many pairwise exclusive events $E_1, E_2, \cdots, E_n$
$\Pr\big(E_1 \vee E_2 \vee \cdots \vee E_n \mid H\big) = \sum_{i=1}^{n} \Pr\big(E_i \mid H\big)$
(finite additivity)

(3) $\Pr\big(E_1 \wedge E_2 \mid H\big) = \Pr\big(E_1 \mid E_2 \wedge H\big) \cdot \Pr\big(E_2 \mid H\big)$
(coherence)

From these axioms the so-called *Bayes' formula* follows:

For any exhaustive and pairwise exclusive finite family of events $E_1, \cdots, E_n$ and arbitrary event E_0 of the event system $\big(E_i,\, i \in I\big)$ the following holds:

$$\Pr\big(E_i \mid E_0 \wedge H\big) = \frac{\Pr\big(E_0 \mid E_i \wedge H\big) \cdot \Pr\big(E_i \mid H\big)}{\sum\limits_{j=1}^{n} \Pr\big(E_0 \mid E_j \wedge H\big) \cdot \Pr\big(E_j \mid H\big)}$$

for $i = 1(1)n$.

Proof: By $E_0 = E_0 \wedge \big(\bigvee_{i=1}^{n} E_i\big) = \bigvee_{i=1}^{n} \big(E_0 \wedge E_i\big)$ we obtain $\Pr\big(E_0 \mid H\big) = \Pr\Big(\bigvee_{i=1}^{n} \big(E_0 \wedge E_i \mid H\big)\Big) = \sum_{i=1}^{n} \Pr\big(E_0 \wedge E_i \mid H\big) = \sum_{i=1}^{n} \Pr\big(E_0 \mid E_i \wedge H\big) \Pr\big(E_i \mid H\big)$.

From the coherence condition we obtain

$\Pr\big(E_0 \wedge E_i \mid H\big) = \Pr\big(E_0 \mid E_i \wedge H\big) \cdot \Pr\big(E_i \mid H\big)$ and $\Pr\big(E_0 \wedge E_i \mid H\big) = \Pr\big(E_i \mid E_0 \wedge H\big) \cdot \Pr\big(E_0 \mid H\big)$ which concludes the proof. //

There are several other theories of probability. For more details compare [1] and [6].

3 Fuzzy probability distributions

More recently looking at histograms for fuzzy data it turns out that frequencies become fuzzy numbers. Therefore it is natural to look for more general concepts of probability, so-called *fuzzy probability distributions*. In this theory probabilities are special *fuzzy numbers*.

A fuzzy number $x^\star$ is characterized by its so-called *characterizing function* $\xi(\cdot)$ which is a generalization of an indicator function $I_A(\cdot)$ of a subset A of the set $\mathbb{R}$ of all real numbers.

A characterizing function $\xi(\cdot)$ is a real function of one real variable x obeying the following:

(1) $0 \le \xi(x) \le 1 \qquad \forall \quad x \in \mathbb{R}$

(2) $\forall\ \delta \in (0;1]$ the so-called δ-cut $C_\delta\big[\xi(\cdot)\big]$, defined by $C_\delta\big[(\cdot)\big] := \big\{x \in \mathbb{R} : \xi(x) \ge \delta\big\}$ is non-empty and a finite union of bounded closed intervals.

In case all δ-cuts are intervals the corresponding fuzzy number is called a *fuzzy interval*.

The system of all fuzzy intervals is denoted by $\mathcal{F}_I(\mathbb{R})$. So-called fuzzy probability distributions $\Pr^\star$ on event systems $\big(E_i,\, i \in I\big)$ are defined in the following way:

A fuzzy probability distribution $\Pr^\star$ is a function $\Pr^\star : \big(E_i,\, i \in I\big) \longrightarrow \mathcal{F}_I(\mathbb{R})$ obeying the following:

(1) $\Pr^\star\big(E_i\big)$ is a fuzzy interval $p^\star$ with characterizing function $\xi_i(\cdot)$ whose support is a subset of $[0;1]$

(2) For all finite families of pairwise exclusive events $E_1, \cdots, E_n$ the following holds true:
Let $C_\delta\big[\Pr^\star(E_i)\big] = \big[a_{i,\delta}; b_{i,\delta}\big]\ \forall\ i = 1(1)n$ and $C_\delta\big[\Pr^\star\big(\bigvee_{i=1}^{n} E_i\big)\big] = \big[c_\delta; d_\delta\big]$ be the corresponding δ-cuts then $c_\delta \ge \sum_{i=1}^{n} a_{i,\delta}$ and $d_\delta \le \sum_{i=1}^{n} b_{i,\delta}$
$\forall\ \delta \in (0;1]$

Special cases of fuzzy probability distributions are defined by so-called *fuzzy densities* $f^\star$ on measure spaces $(M, \mathcal{E}, \mu)$. A fuzzy density on $(M, \mathcal{E}, \mu)$ is a fuzzy valued function $f^\star : M \longrightarrow \mathcal{F}_I\big([0;\infty)\big)$ for which all δ-level functions $\underline{f}_\delta(\cdot)$ and $\overline{f}_\delta(\cdot)$, defined by $C_\delta\big[f^\star(x)\big] = \big[\underline{f}_\delta(x); \overline{f}_\delta(x)\big]\ \forall\, \delta \in (0;1]$, are integrable and there exists a classical probability density $f(\cdot)$ on $\big(M, \mathcal{E}, \mu\big)$, i. e.

$$\int_M f(x)d\mu(x) = 1 \quad \text{for which} \quad \underline{f}_1(x) \le f(x) \le \overline{f}_\delta(x)$$

for all $x \in M$.

Based on fuzzy densities probabilities of classical events $E \in \mathcal{E}$ are defined in the following way:
$\forall\ \delta \in (0;1]$ defining $\mathcal{D}_\delta$ to be the set of all classical probability densities $g(\cdot)$ on $\big(M, \mathcal{E}, \mu\big)$ obeying $\underline{f}_\delta(x) \le g(x) \le \overline{f}_\delta(x)\ \forall\ x \in M$, the fuzzy probability $\Pr^\star(E)$ of an event E is the fuzzy interval $p^\star$ which is generated by the following nested set of closed bounded intervals $\big[a_\delta; b_\delta\big]\ \ \forall\ \delta \in (0;1]$:

$$b_\delta := \sup\Big\{ \int_E g(x)d\mu(x) : g(\cdot) \in \mathcal{D}_\delta \Big\}$$

$$a_\delta := \inf\Big\{ \int_E g(x)d\mu(x) : g(\cdot) \in \mathcal{D}_\delta \Big\}$$

The characterizing function $\psi(\cdot)$ of $p^\star$ is given by its values

$$\psi(x) := \sup\Big\{\delta \cdot I_{[a_\delta; b_\delta]}(x) : \delta \in [0;1]\Big\} \qquad \forall\ x \in \mathbb{R}.$$

This definition yields a fuzzy probability distribution on the events system $\mathcal{E}$ for which the extremal events $\varnothing$ and M have precise probabilities $\Pr^\star(\varnothing) = I_{\{0\}}(\cdot)$ and $\Pr^\star(M) = I_{\{1\}}(\cdot)$. The inequalities for the endpoints of the δ-cuts follow from the integration.

References

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