Robust Equilibria under Linear Tracing Procedure

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Introduction

- Motivated by Ellsberg paradox, **decision theory** has been greatly improved by replacing a single probability distribution with imprecise probabilities (IP) to represent decision makers' uncertainty.
- As an extension of decision theory, **game theory** is concerned with interactive situations (multi-agent decision making). Can game theory be enriched by introducing imprecise probabilities?
- We present a preliminary investigation into the issue by introducing IP into the linear tracing procedure (LTP) proposed by Harsanyi and Selten.

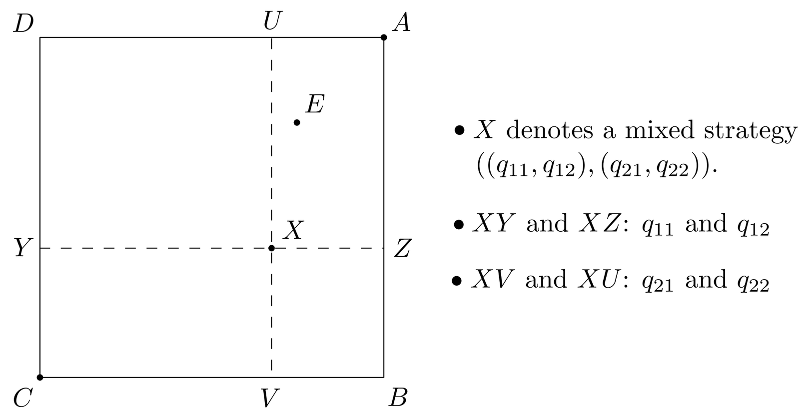
Game-Theoretic Preliminaries

- A finite normal form game $G = \langle I, \{S_i\}, \{u_i\}\rangle_{i\in I}$ consists of:
  - I: a finite set of players who make decisions
  - S_i: a finite set of actions of player i (pure strategies)
  - $u_i : S \to \mathbb{R}$ denotes player i's payoff function, where $S = \prod_{i\in I} S_i$.
- Let Δ_i denote the set of player i's mixed strategies, which can be regarded as probability measures on S_i.

Nash Equilibrium and Its Problem

- **Nash equilibrium** is perhaps the most well-known solution concept for non-cooperative games, which captures the idea that no player has a strict incentive to deviate given the other players' strategies unchanged.
- One problem with NE: There are a variety of nontrivial games that generate (sometimes infinitely) many different Nash equilibria.

	s_{21}	s_{22}
s_{11}	$1,1$	$0,0$
s_{12}	$0,0$	$3,3$



- The game has three Nash equilibrium strategy profiles: $A = (s_{11}, s_{21})$, $C = (s_{12}, s_{22})$, and $E = ((\frac{3}{4}, \frac{1}{4}), (\frac{3}{4}, \frac{1}{4}))$.

Review of LTP

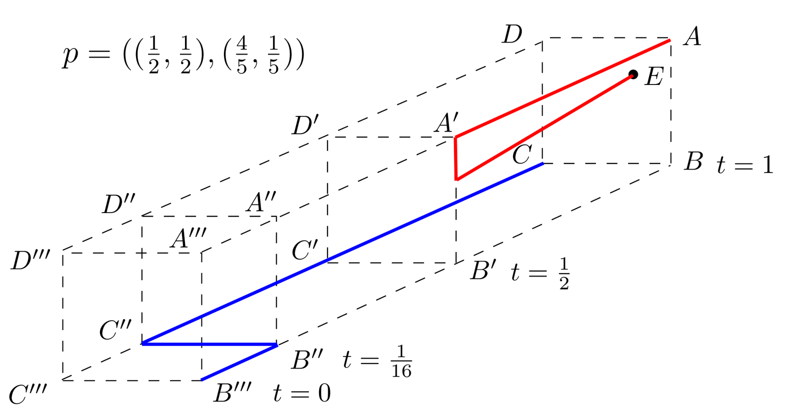
- LTP can be regarded as a rational deliberation process which models how the players gradually update their strategy plans in light of what they know about the opponents' strategic reactions to their own expectations.
- Starting with a common prior distribution, all players gradually change their own tentative strategy plans, as well as their expectations about the other players possible strategies, until they arrive at a certain Nash equilibrium.

Example of LTP

- For a game $G = \langle I, \{S_i\}, \{u_i\}\rangle_{i\in I}$ and a prior $p \in \Delta$, consider a one-parameter family of auxiliary games $\Gamma^{t,p} = \langle I, \{S_i\}, \{u_i^{t,p}\}\rangle_{i\in I}$ with $t \in [0, 1]$, where

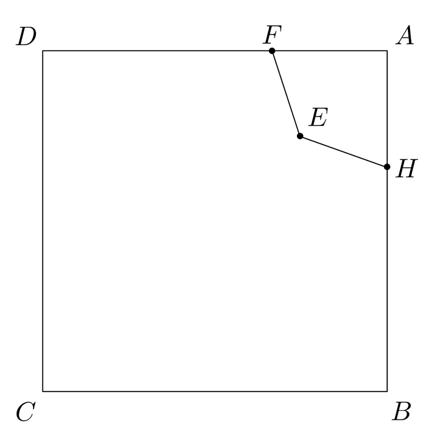
$$u_i^{t,p}(\delta_i, \delta_{-i}) = t \cdot u_i(\delta_i, \delta_{-i}) + (1 - t) \cdot u_i(\delta_i, p_{-i}).$$

- Note that the auxiliary games are solved by considering the solution concept Nash equilibrium as well.
- The graph of LTP starting with the prior $p = ((\frac{1}{2}, \frac{1}{2}), (\frac{4}{5}, \frac{1}{5}))$ is shown to the right.



Source Sets

- **Definition**: For a given game G and a strategy $\delta^* \in NE(G)$, the source set for δ^*, denoted by $\Phi(\delta^*)$, is defined as the set of all prior strategies, based on which the linear tracing procedure yields the Nash equilibrium δ^* as outcome.



Basic Idea

- Note that LTP employs a common prior distribution to represent each player's initial uncertainty about other players' strategy choices.
- It is suggested by the Ellsberg paradox that uncertainty cannot be adequately represented by a single probability distribution and should be expressed by imprecise probabilities, e.g., a set of probabilities.
- Thus, we reexamine LTP by using a (common) set of prior distributions to describe each player's initial beliefs about other players' strategy choices.

Iterative Application of LTP

- Recall that LTP considers a sequence of auxiliary games Γ_p^t to investigate how the equilibria of the original game G behave in these games. LTP should also be applicable to these auxiliary games.
- For each auxiliary game Γ_p^t, consider a new one-parameter class of auxiliary games $\Lambda_p^{t'} = \langle I, \{S_i\}, \{u_i^{t,p}\}\rangle_{i\in I}$ with $t' \in [0, 1]$, where

$$u_i^{t',p}(\delta_i, \delta_{-i}) = t' \cdot u_i^t(\delta_i, \delta_{-i}) + (1 - t') \cdot u_i^t(\delta_i, p_{-i}).$$

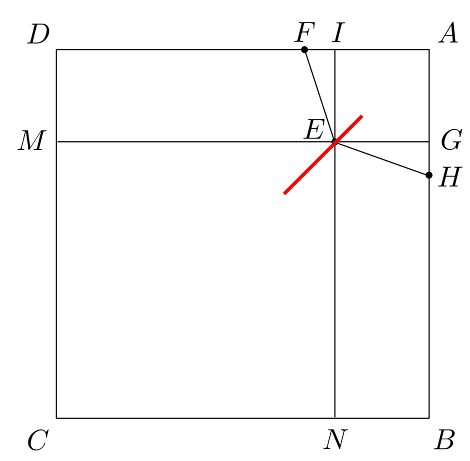
- Clearly, $\Lambda_p^0 = \Gamma_p^0$ and $\Lambda_p^1 = \Gamma_p^t$. Thus, the class of auxiliary games $\Lambda_p^{t'}$ is a subset of the family of auxiliary games Γ_p^t with respect to the game G.

Robustness under LTP

- Let $\Phi^t(\delta^*)$ denote the source set of δ^* with respect to the game Γ_p^t.
- **Definition**: The stability of a prior strategy $p \in \Delta$ w.r.t. δ^* is a real-valued function γ on $\Phi(\delta^*)$, which is defined as $\gamma(p, \delta^*) = 1 - t^*$, where t^* is the smallest t such that $p \in \Phi^t(\delta^*)$.
- **Definition**: Let the players' initial beliefs about the other players' possible behaviors be represented by a set of prior strategies $\mathcal{P}$. The robustness of an equilibrium δ^* w.r.t. $\mathcal{P}$ is defined as $R(\delta^*, \mathcal{P}) = \min_{p\in\mathcal{P}} \gamma(p, \delta^*)$, i.e., the minimum stability index associated with the priors w.r.t. $\mathcal{P}$.

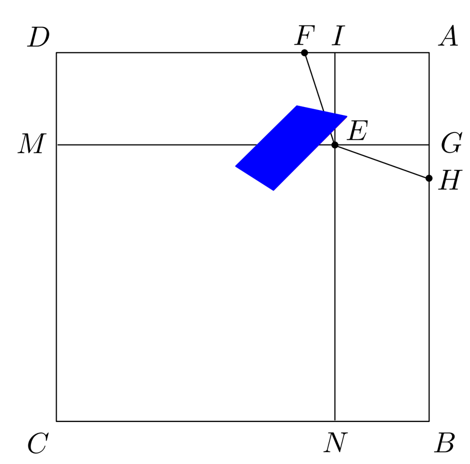
Example: ϵ-contamination under Equilibria Coordination

- Suppose that players' initial belief is represented by the ϵ-contaminated class $\mathcal{P} = \{(1 - \epsilon)P + \epsilon Q, Q \in \mathscr{P}\}$ when $P(E_1) = 0.7$, $P(E_2) = 0.2$, $P(E_3) = 0.1$ and $\epsilon = 0.2$, where $\mathscr{P} = \{Q : Q = p_1E_1 + p_2E_2 + p_3E_3, p_1 + p_2 + p_3 = 1\}$.
- $R(E_1, \mathcal{P}) = R(E_2, \mathcal{P}) = R(E_3, \mathcal{P}) = 1$.



Example: Coordination Failure

- Suppose that all players initially believe that they will mostly choose a strategy from the ϵ-contaminated class $\mathcal{P}$, or otherwise adopt the strategy $D = (s_{12}, s_{21})$ with small probability. All players' initial beliefs are represented by $\mathcal{P}' = \{(1 - \alpha)\mathcal{P} + \alpha D, 0.05 \le \alpha \le 0.2\}$.
- $R(E_1, \mathcal{P}') = 0.8\bar{9}$, $R(E_2, \mathcal{P}') = 0.7\bar{8}$, and $R(E_3, \mathcal{P}') = 0$.



Concluding Remarks

- As a preliminary investigation, we propose a notion of maximin robustness of equilibria by reexamining LTP where players' initial beliefs are represented by a set of probabilities rather a single probability measure.
- In this paper we employ the maximin criterion to define the concept of robustness of equilibria.
  - We have no intention to argue that the maximin rule is the appropriate decision rule under uncertainty.
  - In fact, we intend to consider using some other decision rules like E-admissibility and Maximality to develop solution concepts for games with imprecise probabilities.
- We shall consider developing new solution concepts based on some other game-theoretic solution concepts other than Nash equilibrium by using imprecise probabilities to represent uncertainty in games.