Conglomerable natural extension

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Abstract

We study the weakest conglomerable model that is implied by desirability or probability assessments: the *conglomerable natural extension*.
We show that taking the natural extension of the assessments while imposing conglomerability—the procedure adopted in Walley's theory—does not yield, in general, the conglomerable natural extension (but it does so in the case of the marginal extension). Iterating this process produces a sequence of models that approach the conglomerable natural extension, although it is not known, at this point, whether it is attained in the limit. We give sufficient conditions for this to happen in some special cases, and study the differences between working with coherent sets of desirable gambles and coherent lower previsions.
Our results indicate that it might be necessary to re-think the foundations of Walley's theory of coherent conditional lower previsions for infinite partitions of conditioning events.

Introduction

Coherence for sets of desirable gambles: A set of gambles $\mathcal{R}$ is called coherent when:

(D1) $f \gneq 0 \Rightarrow f \in \mathcal{R}$;

(D2) $0 \notin \mathcal{R}$;

(D3) $f \in \mathcal{R}, \lambda > 0 \Rightarrow \lambda f \in \mathcal{R}$;

(D4) $f, g \in \mathcal{R} \Rightarrow f + g \in \mathcal{R}$.

Conglomerability for sets of gambles: Given a partition $\mathcal{B}$ of Ω, $\mathcal{R}$ is called $\mathcal{B}$-conglomerable when

(D5) $f \neq 0$ and $Bf \in \mathcal{R} \cup \{0\} \forall B \in \mathcal{B} \Rightarrow f \in \mathcal{R}$.

Coherence for lower previsions: A lower prevision $\underline{P}$ on $\mathcal{L}$ is called coherent when:

(C1) $\underline{P}(f) \geq \inf f$ for all $f \in \mathcal{L}$;

(C2) $\underline{P}(\lambda f) = \lambda \underline{P}(f)$ for all $f \in \mathcal{L}$ and $\lambda > 0$;

(C3) $\underline{P}(f+g) \geq \underline{P}(f) + \underline{P}(g)$ for all $f, g \in \mathcal{L}$.

Conglomerability for lower previsions: $\underline{P}$ is called $\mathcal{B}$-conglomerable when

(WC) $(B_n)_n$ pairwise disjoint, $\underline{P}(B_n) > 0$ and $\underline{P}(B_n f) \geq 0 \, \forall n \Rightarrow \underline{P}(\sum_n B_n f) \geq 0$.

Connection

If we make the correspondence

$$\mathcal{R} \leftrightarrow \underline{P}(f) := \sup\{\mu : f - \mu \in \mathcal{R}\}$$

then $\mathcal{R}$ coherent $\Leftrightarrow$ $\underline{P}$ coherent.
However, the conglomerability condition for sets of desirable gambles is stronger than the one for lower previsions:

- $\mathcal{R}$ is $\mathcal{B}$-conglomerable $\Leftrightarrow$ $[\underline{P}(Bf) \geq 0 \, \forall B \in \mathcal{B} \Rightarrow \underline{P}(f) \geq 0]$.
- $\underline{P}$ is $\mathcal{B}$-conglomerable $\Leftrightarrow$ $\left[Bf \in \underline{\mathcal{R}} \cup \{0\} \, \forall B \in \mathcal{B} \Rightarrow f \in \overline{\mathcal{R}},\right]$ where $\underline{\mathcal{R}}$ and $\overline{\mathcal{R}}$ are the interior and closure of $\mathcal{R}$ in the topology of uniform convergence.

Congnatex for sets of gambles

Definition. Let $\mathcal{R}$ be a coherent set of gambles. The smallest superset $\mathcal{F}$ that satisfies (D1)–(D5) with respect to a fixed partition $\mathcal{B}$ is called the $\mathcal{B}$-conglomerable natural extension of $\mathcal{R}$.

Approximation by a sequence. Let us define

$$\mathcal{R}^* := \{f \neq 0 \colon (\forall B \in \mathcal{B}) Bf \in \mathcal{R} \cup \{0\}\}$$
$$\mathcal{E}_1 := \mathcal{R} \oplus \mathcal{R}^*$$

and for all $n \geq 2$, let

$$\mathcal{E}_{n-1} := \{f \neq 0 \colon (\forall B \in \mathcal{B}) Bf \in \mathcal{E}_{n-1} \cup \{0\}\}$$
$$\mathcal{E}_n := \mathcal{E}_{n-1} \oplus \mathcal{E}^*_{n-1}.$$

- $\mathcal{E}_n$ satisfies (D1)–(D4) and $\mathcal{E}^*_n$ satisfies (D1)–(D5) for every n.
- $\cup_n \mathcal{E}_n \subseteq \mathcal{F}$.
- $\mathcal{F} = \mathcal{E}_n \Leftrightarrow \mathcal{E}^*_n \subseteq \mathcal{E} \Leftrightarrow \mathcal{E}_n$ conglomerable.
- $\mathcal{R}^* = \mathcal{F} \Leftrightarrow \mathcal{R} \subseteq \mathcal{R}^*$.
- In general $\mathcal{E}_n \nsubseteq \overline{\mathcal{E}_{n-1}}$.

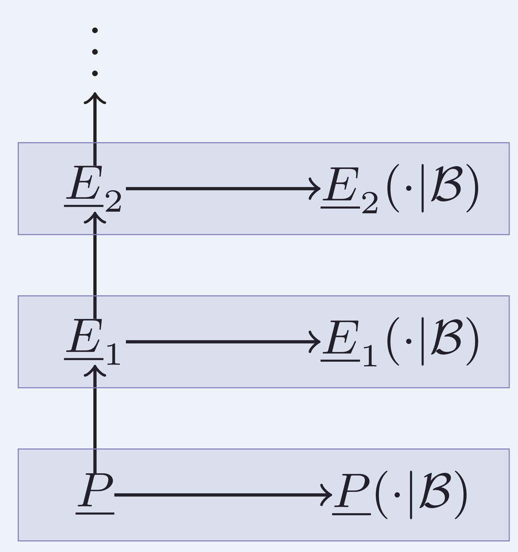
Congnatex for lower previsions

Definition. Similarly, given a coherent lower prevision $\underline{P}$ on $\mathcal{L}$ its $\mathcal{B}$-conglomerable natural extension is the smallest coherent lower prevision $\underline{F}$ that dominates $\underline{P}$ and is $\mathcal{B}$-conglomerable.

Connection with Walley's natural extension. Let $\underline{P}(\cdot|\mathcal{B})$ be the conditional natural extension of $\underline{P}$, and $\underline{E}$ the natural extension of $\underline{P}, \underline{P}(\cdot|\mathcal{B})$.

- $\underline{E} \leq \underline{F}$, but they do not coincide in general.
- $\underline{E} = \underline{F} \Leftrightarrow \underline{E}, \underline{E}(\cdot|\mathcal{B})$ are coherent.
- Given $\underline{Q} = \underline{P}(\underline{P}(\cdot|\mathcal{B}))$, it holds that $\mathcal{M}(\underline{E}) = \mathcal{M}(\underline{P}) \cap \mathcal{M}(\underline{Q})$.
- $\underline{E} = \underline{F}$ under any of the following conditions:
  - $\underline{P}(\cdot|\mathcal{B})$ is linear and $\underline{F}$ exists,
  - There exists $\underline{P}' \geq \underline{P}$ coherent with $\underline{P}(\cdot|\mathcal{B})$,
  - $\underline{P}(\cdot|\mathcal{B})$ vacuous,
  - $\underline{E}(\cdot|\mathcal{B}) = \underline{P}(\cdot|\mathcal{B})$.

Approximation by a sequence. We can approximate $\underline{F}$ using the following construction:



where $\rightarrow$ applies the conditional natural extension and $\uparrow$ the unconditional natural extension.

- $\underline{E}_n \leq \underline{F} \, \forall n$, and $\underline{E}_n \neq \underline{E}_{n+1}$ unless $\underline{E}_n = \underline{F}$.

Relationship

Let $\mathcal{R}$ be a coherent set of desirable gambles and $\underline{P}$ its associated lower prevision. Let $(\mathcal{E}_n)_n$ and $(\underline{E}_n)_n$ be the approximating sequences of the conglomerable natural extensions.

- $\mathcal{E}_1 = \mathcal{F} \Rightarrow \underline{E}_1 = \underline{F}$, *but the converse is not true.*
- Let $(\underline{P}_n)_n$ be the sequence of coherent lower previsions associated to $(\mathcal{E}_n)_n$. Then $\underline{E}_n \leq \underline{P}_n$ for every n, and they coincide if $\underline{P}(B) > 0 \, \forall B \in \mathcal{B}$.

Several partitions

Next we consider a finite number of sets $\mathcal{R}_1, \dots, \mathcal{R}_m$, where $\mathcal{R}_i$ satisfies (D1)–(D5) with respect to a partition $\mathcal{B}_i$, and we look for the smallest superset $\mathcal{F}$, if it exists, that satisfies (D1)–(D5) with respect to $\{\mathcal{B}_1, \dots, \mathcal{B}_m\}$.

Characterisation. Conglomerability with respect to $\mathcal{B}_1, \dots, \mathcal{B}_m$ is equivalent to conglomerability with respect to all partitions $\mathcal{B}$ such that

$$(\forall B \in \mathcal{B})(\exists j \in \{1, \dots, m\}) \text{ s.t. } B \in \mathcal{B}_j.$$

Measurable gambles. Given a partition $\mathcal{B}$ of Ω, a gamble f on Ω is called $\mathcal{B}$-measurable when it is constant on the elements of $\mathcal{B}$. The set of all $\mathcal{B}$-measurable gambles is denoted by $\mathcal{G}(\mathcal{B})$. $\mathcal{R}$ is called coherent relative to $\mathcal{G}(\mathcal{B})$ if it satisfies (D2)–(D4) and

(D1*) if $f \in \mathcal{G}(\mathcal{B})$ and $f \gneq 0$ then $f \in \mathcal{R}$.

Marginal extension theorem. Assume that $\mathcal{B}_{i+1}$ is finer than $\mathcal{B}_i$ for $i = 1, \dots, n-1$. Let $\mathcal{R}_0$ be a set of desirable gambles coherent relative to $\mathcal{G}(\mathcal{B}_1)$. For each $i = 1, \dots, n-1$ and each $B_i \in \mathcal{B}_i$, let

$$\mathcal{B}_{i+1}|B_i := \{B_{i+1} \in \mathcal{B}_{i+1} \colon B_{i+1} \subseteq B_i\}$$

and $\mathcal{R}_i|B_i$ be a coherent set of desirable gambles on $\mathcal{L}(B_i)$ relative to $\mathcal{G}(\mathcal{B}_{i+1}|B_i)$. Finally, for each $B_n \in \mathcal{B}_n$, let $\mathcal{R}_n|B_n$ be a coherent set of desirable gambles on $\mathcal{L}(B_n)$.
The conglomerable natural extension $\mathcal{F}$ of $\mathcal{R}_0$ and $\mathcal{R}_i|B_i$, $B_i \in \mathcal{B}_i$, is given by

$$\left\{ f_0 + \sum_{i=1}^{n} \sum_{B_i \in \mathcal{B}_i} B_i g_{B_i} \colon f_0 \in \mathcal{R}_0 \cup \{0\}, g_{B_i} \in \mathcal{R}_i|B_i \cup \{0\} \right\} \setminus \{0\}.$$

The case of lower previsions. $\underline{P}$ is $\mathcal{B}_i$-conglomerable for all $i = 1, \dots, m \Leftrightarrow$ there are $\underline{P}_1(\cdot|\mathcal{B}_1), \dots, \underline{P}_m(\cdot|\mathcal{B}_m)$ weakly coherent with $\underline{P}$.

Essential references

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