A fully polynomial time approximation scheme for updating credal networks of bounded treewidth and number of variable states

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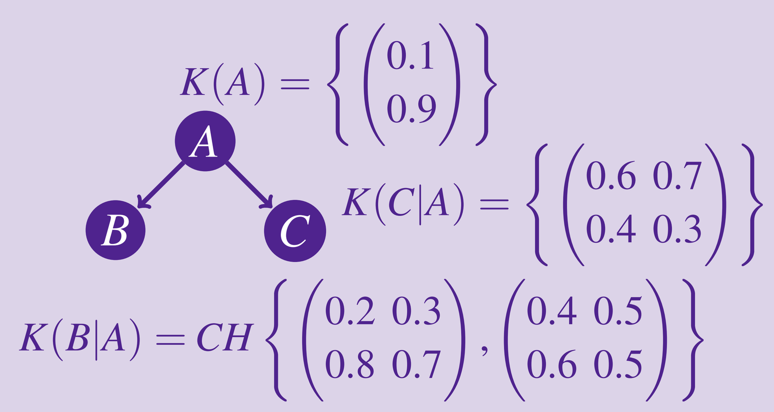
Credal networks

An extensively specified credal network is a pair $(\mathscr{G},\mathbb{K})$, where $\mathbb{K}$ is a collection of finitely generated closed convex extensive credal sets $K(X|\mathrm{pa}(X))$. We assume each extensive set $K(X|\mathrm{pa}(X))$ is represented by a finite number of conditional probability tables $P(X|\mathrm{pa}(X))$.



Exact inference

Belief updating consists in computing upper and lower bounds of the posterior probability of a query given evidence. For instance, the upper posterior probability is given by

$$\overline{P}(q|e)=\max\left\{\frac{\sum_X \mathbb{I}(q,e)\prod_{i=1}^n P(X_i|\mathrm{pa}(X_i))}{\sum_X \mathbb{I}(e)\prod_{i=1}^n P(X_i|\mathrm{pa}(X_i))} : P(X_i|\mathrm{pa}(X_i))\in \mathtt{ext}[K(X_i|\mathrm{pa}(X_i))]\right\}$$

Let

$$K_{\max}(Q,E)=\left\{P(Q,E)\in K(Q,E) : \nexists R(Q,E)\in K(Q,E) \text{ s.t. } R(q,e)\geq P(q,e) \text{ and } \sum_{q'\neq q} R(q',e)\leq \sum_{q'\neq q} P(q',e)\right\}$$

It can be shown that

$$\overline{P}(q|e)=\max\left\{\left(1+\frac{\sum_{q'\neq q}P(q',e)}{P(q,e)}\right)^{-1} : P(Q,E)\in K_{\max}(Q,E)\right\}$$

Since $K_{\max}(Q,E)$ is usually small, we can compute $\overline{P}(q|e)$ by enumeration. We use a message-passing scheme to obtain $K_{\max}(Q,E)$.

Provably good approximate inference

For any given $\varepsilon>1$, a fully polynomial time approximation scheme returns in time polynomial in the input a solution $P_\varepsilon(q|e)$ such that $\varepsilon P_\varepsilon(q|e)\geq \overline{P}(q,e)$.

Let

$$K_\varepsilon(Q,E)=\left\{P(Q,E)\in K(Q,E) : \nexists R(Q,E)\in K(Q,E) \text{ s.t. } \varepsilon' R(q,e)\geq P(q,e) \text{ and } \sum_{q'\neq q} R(q',e)\leq \varepsilon' \sum_{q'\neq q} P(q',e)\right\}$$

Then we can obtain an ε-solution by

$$P_\varepsilon(q|e)=\max\left\{\left(1+\frac{\sum_{q'\neq q}P(q',e)}{P(q,e)}\right)^{-1} : P(Q,E)\in K_\varepsilon(Q,E)\right\}$$

Likewise the exact case, we can compute $K_\varepsilon(Q,E)$ with a message-passing scheme. If the network has bounded treewidth and the variables have bounded cardinality, then we show that $K_\varepsilon(Q,E)$ has bounded cardinality, and can be computed in polynomial time.

Valuation algebra

A valuation ϕ over a set of variables X is a set of pairs (f,g) of functions f and g with domain X and image $[0,1]$.

We assume two operations over valuations:

- Combination takes two valuations ϕ and ψ and returns a valuation $\phi\otimes\psi$;
- Projection takes a valuation ϕ over X and set of variables Y and outputs a valuation over $\phi^{\downarrow Y}$ over Y.

Exact operations

Consider two valuations ϕ and ψ and a set of variables X.

Set-combination:

$$\phi=\{(f\cdot g, f'\cdot g') : (f,f')\in\phi, (g,g')\in\psi\}$$

Set-projection:

$$\phi^{\downarrow X}=\left\{\left(\sum_{-X} f, \sum_{-X} f'\right) : (f,f')\in\phi\right\}$$

Max-Combination:

$$\phi\oplus\psi=\{(f,f')\in\phi\times\psi : \nexists (g,g')\neq (f,f') \text{ s.t. } g\leq f \text{ and } g'\geq f'\}$$

Max-projection:

$$\phi^{\Downarrow X}=\left\{(f,f')\in\phi^{\downarrow X} : \nexists (g,g')\neq (f,f') \text{ s.t. } g\leq f \text{ and } g'\geq f'\right\}$$

Approximate operations

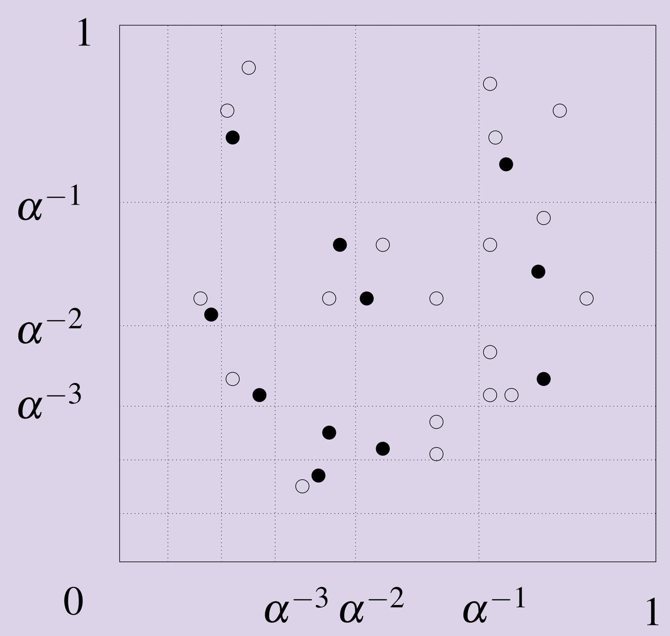
For $\alpha>1$, say that pairs (f,f') and (g,g') are α-equivalent, and write $(f,f')\equiv(g,g')$, if for all x one of the following

- $f(x)=g(x)$
- $f(x)>0, g(x)>0$ and $\lfloor\log_\alpha f(x)\rfloor=\lfloor\log_\alpha g(x)\rfloor$

and one of the following

- $f'(x)=g'(x)$
- $f'(x)>0, g'(x)>0$ and $\lfloor\log_\alpha f'(x)\rfloor=\lfloor\log_\alpha g'(x)\rfloor$

hold.

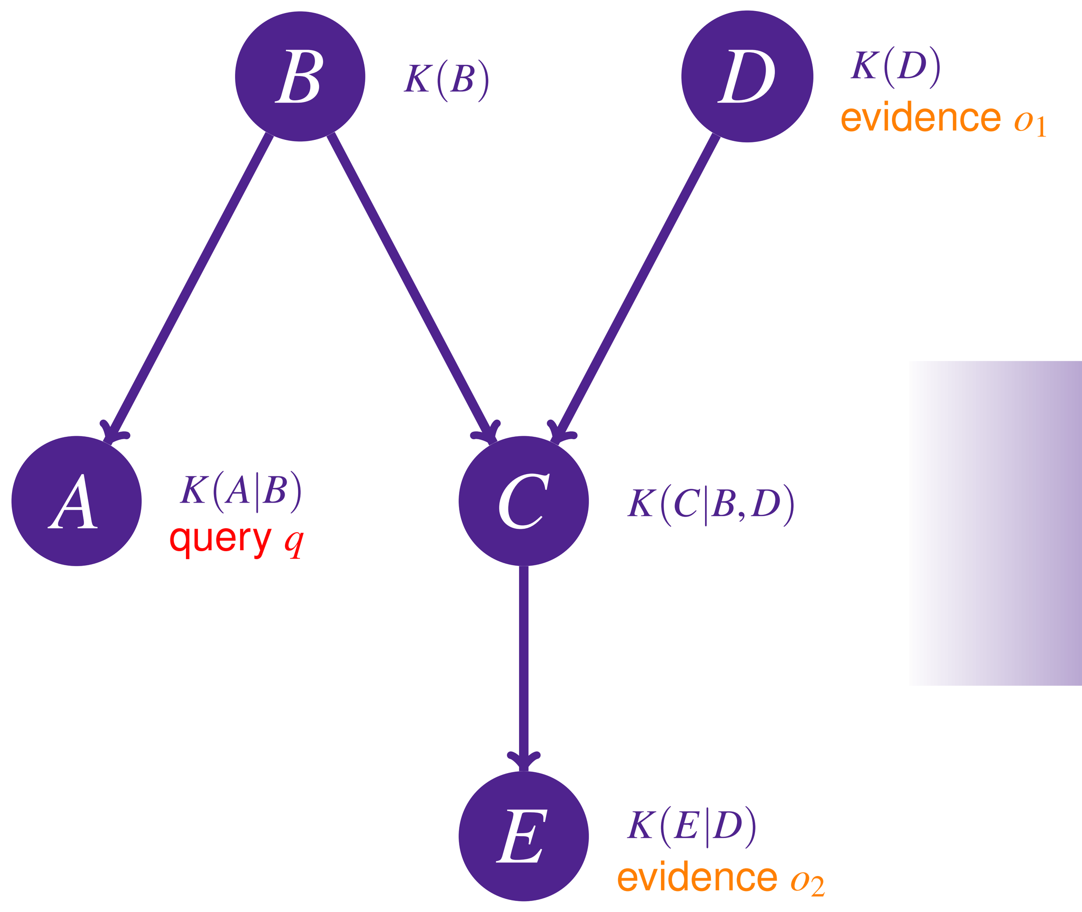


Given valuations ϕ and ψ, their α-combination is defined as

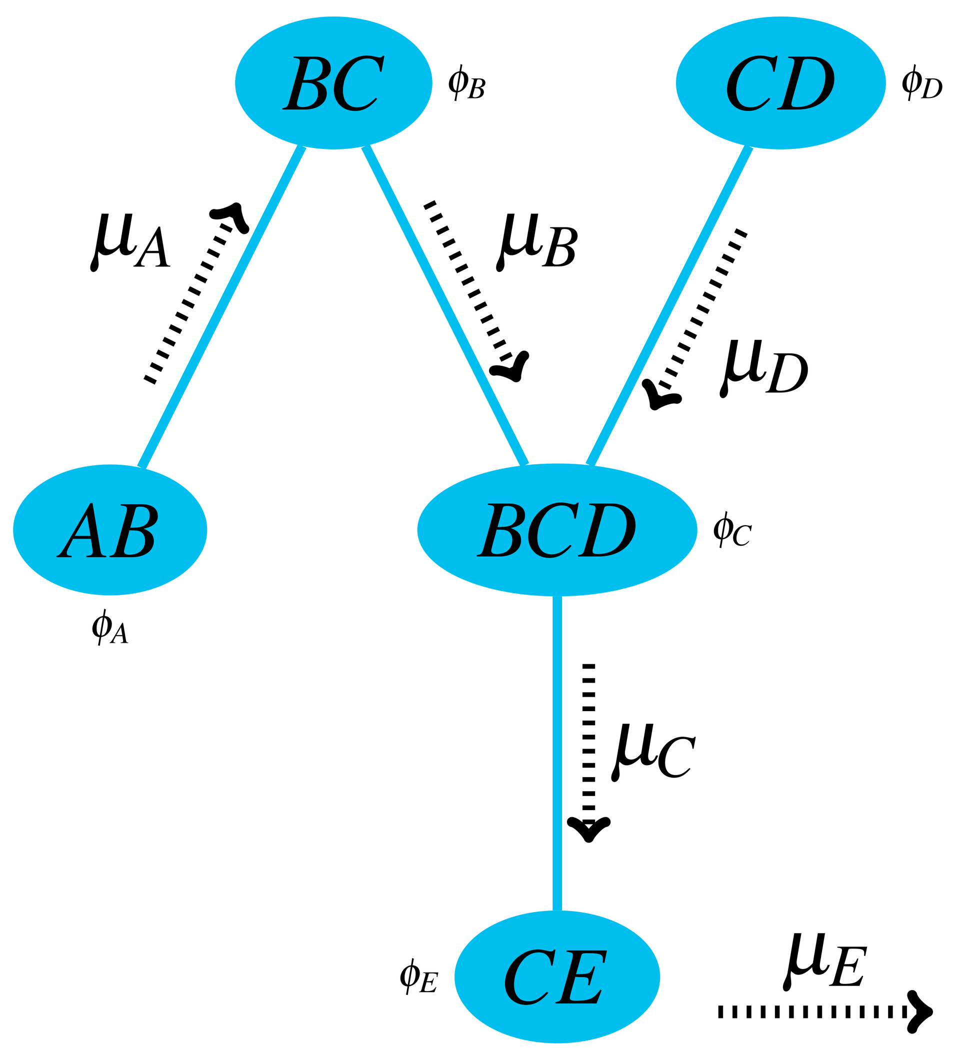
$$\phi\oplus_\alpha\psi=\{(f,f')\in\phi\times\psi : \nexists(g,g')\in\phi\times\psi \text{ s.t. } (g,g')\equiv(f,f')\}$$

Example

Credal network



Join tree



Initialization

$$\phi_A=\left\{\left(\sum_{a\neq q}P(a|B),P(q|B)\right) : P(A|B)\in\mathtt{ext}[K(A|B)]\right\}$$

$$\phi_B=\{(P(B),P(B)) : P(B)\in\mathtt{ext}[K(B)]\}$$

$$\phi_D=\{(P(o_1),P(o_1)) : P(D)\in\mathtt{ext}[K(D)]\}$$

$$\phi_C=\{(P(C|B,o_1),P(C|B,o_1)) : P(C|B,D)\in\mathtt{ext}[K(C|B,D)]\}$$

$$\phi_E=\{(P(o_2|C),P(o_1|C)) : P(E|C)\in\mathtt{ext}[K(E|C)]\}$$

Propagation

$$\mu_A=\phi_A \qquad \mu_B=[\mu_A\otimes\phi_B]^{\downarrow B}$$

$$\mu_D=\phi_D \qquad \mu_C=[\mu_B\otimes\mu_D\otimes\phi_C]^{\downarrow C}$$

$$\mu_E=[\mu_C\otimes\phi_E]^{\downarrow\emptyset}=\left\{\left(\sum_{a\neq q}P(a,o_1,o_2),P(q,o_1,o_2)\right)\right\}$$

Termination

Return $\max\{(1+f'/f)^{-1} : (f,f')\in\mu_E\}=P(q|o_1,o_2)$