Conditioning, Conditional Independence and Irrelevance in Evidence Theory

Jiřina Vejnarová
Institute of Information Theory and Automation of the ASCR
vejnar@utia.cas.cz

Independence

Let m be a basic assignment on $\mathbf{X}_N$ and $K, L \subset N$ be disjoint. We say that groups of variables X_K and X_L are *independent with respect to basic assignment m* (and denote it by $K \perp\!\!\!\perp L\ [m]$) if

$$m^{\downarrow K\cup L}(A) = m^{\downarrow K}(A^{\downarrow K}) \cdot m^{\downarrow L}(A^{\downarrow L})$$

for all $A \subseteq \mathbf{X}_{K\cup L}$ for which $A = A^{\downarrow K} \times A^{\downarrow L}$, and $m(A) = 0$ otherwise.

Irrelevance

Group of variables X_L is *irrelevant* to X_K ($K \cap L = \emptyset$) if for any $B \subseteq \mathbf{X}_L$ such that $Pl(B) > 0$ (or $Bel(B) > 0$)

$$m_{X_K|X_L}(A|B) = m(A)$$

for any $A \subseteq \mathbf{X}_K$.

Conditioning rules

Dempster's rule of conditioning

$$m(A|B) = \frac{\sum_{C\subseteq \mathbf{X}_N: C\cap B = A} m(C)}{Pl(B)}$$

for any $\emptyset \neq A \subseteq \mathbf{X}_N$, $B \subseteq \mathbf{X}_N$ such that $Pl(B) > 0$, $m(\emptyset|B) = 0$.

Focusing

$$m(A||B) = \begin{cases} \dfrac{m(A)}{Bel(B)} & \text{if} \quad A \subseteq B, \\ 0 & \text{otherwise.} \end{cases}$$

$B \subseteq \mathbf{X}_N$ such that $Bel(B) > 0$.

Properties

Let X_K and X_L ($K \cap L = \emptyset$) be independent groups of variables (under joint basic assignment m defined on $\mathbf{X}_{K\cup L}$). Then X_L are irrelevant to X_K with respect to both Dempster's conditioning rule and focusing.

- Irrelevance with respect to Dempster's rule of conditioning (DRC) does not imply that with respect to focusing. (Example 1)
- Irrelevance with respect to DRC does not imply independence. (Example 1)
- Irrelevance with respect to DRC is not symmetric, in general.
- Even in case of symmetry it does not imply independence. (Example 1)
- The same results hold also for focusing.

Example 1

Let X_1 and X_2 be two binary variables (with values in $\mathbf{X}_i = \{a_i, \bar{a}_i\}$) with joint basic assignment m defined as follows:

$$m(\{(a_1, a_2)\}) = \frac{1}{2},$$
$$m(\mathbf{X}_1 \times \mathbf{X}_2 \setminus \{(\bar{a}_1, \bar{a}_2)\}) = m(\mathbf{X}_1 \times \mathbf{X}_2) = \frac{1}{4},$$

From these values one can obtain

$$m^{\downarrow 2}(\{a_2\}) = m^{\downarrow 2}(\mathbf{X}_2) = \frac{1}{2},$$

and therefore

$$Bel^{\downarrow 2}(\{a_2\}) = \frac{1}{2}, \qquad Bel^{\downarrow 2}(\{\bar{a}_2\}) = 0.$$
$$Pl^{\downarrow 2}(\{a_2\}) = 1, \qquad Pl^{\downarrow 2}(\{\bar{a}_2\}) = \frac{1}{2}.$$

Computing conditional basic assignments (according to Dempster's conditioning rule) one can easily see that

$$m_{X_1|X_2}(\{a_1\}|\{a_2\}) = m_{X_1|X_2}(\{a_1\}|\{\bar{a}_2\}) = \frac{1}{2} = m^{\downarrow 1}(\{a_1\}),$$
$$m_{X_1|X_2}(\{\bar{a}_1\}|\{a_2\}) = m_{X_1|X_2}(\{\bar{a}_1\}|\{\bar{a}_2\}) = 0 = m^{\downarrow 1}(\{\bar{a}_1\}),$$
$$m_{X_1|X_2}(\mathbf{X}_1|\{a_2\}) = m_{X_1|X_2}(\mathbf{X}_1|\{\bar{a}_2\}) = \frac{1}{2} = m^{\downarrow 1}(\mathbf{X}_1),$$

i.e. X_2 is irrelevant to X_1 (with respect to Dempster's conditioning rule). On the other hand, as e.g.

$$m_{X_1||X_2}((\{a_1\}||(\{a_2\})) = \frac{m(\{(a_1, a_2)\}))}{Bel(\{a_2\})} = 1 \neq \frac{1}{2} = m^{\downarrow 1}(\{a_1\}),$$

they are not irrelevant with respect to focusing. Furthermore, although also X_1 is irrelevant to X_2, $X_!$ and X_2 are not independent, as $\mathbf{X}_1 \times \mathbf{X}_2 \setminus \{(\bar{a}_1, \bar{a}_2)\}$ is not a rectangle.

Conditional independence

Let m be a basic assignment on $\mathbf{X}_N$ and $K, L, M \subset N$ be disjoint, $K \neq \emptyset \neq L$. We say that groups of variables X_K and X_L are *conditionally independent given X_M with respect to m* (and denote it by $K \perp\!\!\!\perp L|M\ [m]$), if the equality

$$m^{\downarrow K\cup L\cup M}(A) \cdot m^{\downarrow M}(A^{\downarrow M}) = m^{\downarrow K\cup M}(A^{\downarrow K\cup M}) \cdot m^{\downarrow L\cup M}(A^{\downarrow L\cup M})$$

holds for any $A \subseteq \mathbf{X}_{K\cup L\cup M}$ such that $A = A^{\downarrow K\cup M} \bowtie A^{\downarrow L\cup M}$, and $m(A) = 0$ otherwise.

Conditional irrelevance

Group of variables X_L is *conditionally irrelevant* to X_K given X_M (K, L, M disjoint, $K \neq \emptyset \neq L$) if for any $B \subseteq \mathbf{X}_{L\cup M}$ such that $Pl(B) > 0$ ($Bel(B) > 0$, respectively)

$$m_{X_K|X_L X_M}(A|B) = m_{X_K|X_M}(A|B^{\downarrow M}).$$

Properties

Conditional independence does not imply conditional irrelevance either for Dempster's conditioning rule or focusing. (Example 2)

Let X_K and X_L be conditionally independent groups of variables given X_M under joint basic assignment m on $\mathbf{X}_{K\cup L\cup M}$ (K, L, M disjoint, $K \neq \emptyset \neq L$). Then

$$m_{X_K||X_L X_M}(A||B) = m_{X_K||X_M}(A||B^{\downarrow M})$$

for any $m^{\downarrow L\cup M}$-atom $B \subseteq \mathbf{X}_{L\cup M}$ such that $B^{\downarrow M}$ is $m^{\downarrow M}$-atom and $A \subseteq \mathbf{X}_K$.

Example 2

Let X_1, X_2 and X_3 be three variables with values in $\mathbf{X}_1, \mathbf{X}_2$ and $\mathbf{X}_3$ respectively, $\mathbf{X}_i = \{a_i, \bar{a}_i\}, i = 1, 2, 3$, and their joint basic assignment is defined as follows:

$$m(\{(x_1, x_2, x_3)\}) = \tfrac{1}{16},\ m(\mathbf{X}_1 \times \mathbf{X}_2 \times \mathbf{X}_3) = \tfrac{1}{2},$$

for $x_i = a_i, \bar{a}_i$, values of m on the remaining sets being 0, i.e. we have 9 focal elements — 8 singletons and the whole frame of discernment. Its marginal basic assignments on $\mathbf{X}_1 \times \mathbf{X}_3, \mathbf{X}_2 \times \mathbf{X}_3$ and $\mathbf{X}_3$ are

$$m^{\downarrow 13}(\{(x_1, x_3)\}) = \tfrac{1}{8},\ m^{\downarrow 13}(\mathbf{X}_1 \times \mathbf{X}_3) = \tfrac{1}{2},$$
$$m^{\downarrow 23}(\{(x_2, x_3)\}) = \tfrac{1}{8},\ m^{\downarrow 23}(\mathbf{X}_2 \times \mathbf{X}_3) = \tfrac{1}{2},$$
$$m^{\downarrow 3}(\{x_3\}) = \tfrac{1}{4}, \qquad m^{\downarrow 3}(\mathbf{X}_3) = \tfrac{1}{2},$$

respectively (values of m of remaining subsets being 0, again). It is easy (but somewhat time-consuming) to show that for any $A \subseteq \mathbf{X}_1 \times \mathbf{X}_2 \times \mathbf{X}_3$ such that $A = A^{\downarrow 13} \bowtie A^{\downarrow 23}$

$$m(A) \cdot m^{\downarrow 3}(A^{\downarrow 3}) = m^{\downarrow 13}(A^{\downarrow 13}) \cdot m^{\downarrow 23}(A^{\downarrow 23}),$$

the values of remaining sets being zero, i.e. $\{1\} \perp\!\!\!\perp \{2\}|\{3\}\ [m]$ holds.
Now, let us show, that X_2 is not irrelevant to X_1 given X_3. To do so, we have to compute $m_{X_1|X_2X_3}$ and $m_{X_1|X_3}$. First, let us take into account that

$$Pl(\{(x_2, x_3)\}) = \tfrac{5}{8},\ Pl(\{x_3\}) = \tfrac{3}{4}$$

for any $x_i = a_i, \bar{a}_i, i = 2, 3$ and that $(\mathbf{X}_1 \times \mathbf{X}_2 \times \mathbf{X}_3 \cap \{(a_2, a_3)\}^{\uparrow 123})^{\downarrow 1} = \mathbf{X}_1$ and similarly $(\mathbf{X}_1 \times \mathbf{X}_3 \cap \{a_3\}^{\uparrow 13})^{\downarrow 1} = \mathbf{X}_1$. Then we have

$$m_{X_1|X_2X_3}(\{x_1\}|\{(a_2, a_3)\}) = \frac{m(\{(x_1, a_2, a_3)\})}{Pl(\{(a_2, a_3)\})} = \frac{1}{10} \text{ and } m_{X_1|X_2X_3}(\mathbf{X}_1|\{(a_2, a_3)\}) = \frac{m(\mathbf{X}_1 \times \mathbf{X}_2 \times \mathbf{X}_3)}{Pl(\{(a_2, a_3)\})} = \frac{4}{5},$$

while

$$m_{X_1|X_3}(\{x_1\}|\{a_3\}) = \frac{m(\{(x_1, a_3)\})}{Pl(\{a_3\})} = \frac{1}{6} \text{ and } m_{X_1|X_3}(\mathbf{X}_1|\{a_3\}) = \frac{m(\mathbf{X}_1 \times \mathbf{X}_3)}{Pl(\{a_3\})} = \frac{2}{3},$$

i.e. $m_{X_1|X_2X_3} \neq m_{X_1|X_3}$. In other words X_2 is not conditionally irrelevant to X_2 given $X_3 = a_3$ (with respect to Dempster's conditioning rule).
To show an analogous result for focusing we need to condition by a non-atom. Let us consider a set $B = \{(a_2, a_3), (\bar{a}_2, \bar{a}_3)\} \subseteq \mathbf{X}_2 \times \mathbf{X}_3$. One can easily compute that $Bel(B) = \frac{1}{4}$ and therefore

$$m_{X_1|X_2X_3}(\{x_1\}|B) = \frac{m(\{x_1, a_2, a_3\}) + m(\{x_1, \bar{a}_2, \bar{a}_3\})}{Bel(B)} = \frac{1}{2} \text{ and } m_{X_1|X_2X_3}(\mathbf{X}_1|B) = 0,$$

while,

$$m_{X_1|X_3}(\{x_1\}|B^{\downarrow 3}) = m^{\downarrow 1}(\{a_1\}) = \frac{1}{4} \text{ and } m_{X_1|X_2X_3}(\mathbf{X}_1|B^{\downarrow 3}) = m^{\downarrow 1}(\mathbf{X}_1) = \frac{1}{2}.$$

as $B^{\downarrow 3} = \mathbf{X}_3$. It means that conditional independence does not imply either conditional irrelevance with respect to focusing.