Non-conflicting and Conflicting Parts of Belief Functions

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1 Introduction

Belief functions are one of the widely used formalisms for uncertainty representation and processing that enables representation of incomplete and uncertain knowledge, belief updating, and combination of evidence. They were originally introduced as a principal notion of the Dempster-Shafer Theory or the Mathematical Theory of Evidence [17].

When combining belief functions (BFs) by the conjunctive rules of combination, conflicts often appear, which are assigned to $\emptyset$ by un-normalized conjunctive rule $\odot$ or normalized by Dempster's rule of combination $\oplus$. Combination of conflicting BFs and interpretation of conflicts is often questionable in real applications, thus a series of alternative combination rules was suggested and a series of papers on conflicting belief functions was published, e.g. [2, 5, 16, 19].

In IPMU 2010 [9], new ideas concerning interpretation, definition, and measurement of conflicts of BFs were introduced. We presented three new approaches to interpretation and computation of conflicts: combinational conflict, plausibility conflict, and comparative conflict. **Differences were made between conflicts between BFs and internal conflicts of single BF; a conflict between BFs was distinguished from the difference between BFs.**

When analyzing mathematical properties of the three approaches to conflicts of BFs in [10], there appears **a possibility of expression of a BF Bel as Dempster's sum of non-conflicting BF Bel_0 with the same plausibility decisional support as the original BF Bel has and of indecisive BF Bel_S which does not prefer any of the elements of frame of discernment. The presented contribution analyses existence and uniqueness of such BFs Bel_0 and Bel_S.**

2 Preliminaries

General Primer on Belief Functions

We assume classic definitions of basic notions from theory of *belief functions* (BFs) [17] on finite frames of discernment $\Omega_n = \{\omega_1, \omega_2, ..., \omega_n\}$. A **basic belief assignment (bba)** is a mapping $m : \mathcal{P}(\Omega) \longrightarrow [0,1]$ such that $\sum_{A\subseteq\Omega} m(A) = 1$; the values of the bba are called ***basic belief masses (bbm)***. $m(\emptyset) = 0$ is usually assumed, then we speak about *normalized bba*. A ***belief function (BF)*** is a mapping $Bel : \mathcal{P}(\Omega) \longrightarrow [0,1]$, $Bel(A) = \sum_{\emptyset\neq X\subseteq A} m(X)$. A ***plausibility function*** $Pl(A) = \sum_{\emptyset\neq A\cap X} m(X)$. There is a **unique correspondence** among m and related Bel and Pl thus we often speak about m as about BF.

A ***focal element*** is a subset X of the frame of discernment, such that $m(X) > 0$. If all the focal elements are ***singletons*** (i.e. one-element subsets of Ω), then we speak about a ***Bayesian belief function*** (BBF), it is a probability distribution on Ω in fact. If all the focal elements are either singletons or whole Ω (i.e. $|X| = 1$ or $|X| = |\Omega|$), then we speak about a ***quasi-Bayesian belief function*** (qBBF), it is something like 'un-normalized probability distribution'. If all focal elements are nested, we speak about ***consonant belief function***.

Dempster's (conjunctive) rule of combination $\oplus$ is $(m_1 \oplus m_2)(A) = \sum_{X\cap Y=A} K m_1(X)m_2(Y)$ for $A \neq \emptyset$, where $K = \frac{1}{1-\kappa}$, $\kappa = \sum_{X\cap Y=\emptyset} m_1(X)m_2(Y)$, and $(m_1 \oplus m_2)(\emptyset) = 0$, see [17]; putting $K = 1$ and $(m_1 \oplus m_2)(\emptyset) = \kappa$ we obtain the ***un-normalized conjunctive rule of combination*** $\odot$, [18].
The disjunctive rule of combination is given by the formula $(m_1 \circledcirc m_2)(A) = \sum_{X\cup Y=A} m_1(X)m_2(Y)$ [12].
Yager's rule of combination $\circledY$, see [21], is given as $(m_1 \circledY m_2)(\emptyset) = 0$, $(m_1 \circledY m_2)(\Theta) = m_1(\Theta)m_2(\Theta) + \sum_{X,Y\subseteq\Theta,\, X\cap Y=\emptyset} m_1(X)m_2(Y)$, and $(m_1 \circledY m_2)(A) = \sum_{X,Y\subseteq\Theta,\, X\cap Y=A} m_1(X)m_2(Y)$ for $\emptyset \neq A \subset \Theta$.
Dubois-Prade's rule of combination $\circledD$ is given as $(m_1 \circledD m_2)(A) = \sum_{X,Y\subseteq\Theta,\, X\cap Y=A} m_1(X)m_2(Y) + \sum_{X,Y\subseteq\Theta,\, X\cap Y=\emptyset, X\cup Y=A} m_1(X)m_2(Y)$ for $\emptyset \neq A \subseteq \Theta$, and $(m_1 \circledD m_2)(\emptyset) = 0$, see [11].

We say that BF Bel is ***non-conflicting*** when conjunctive combination of Bel with itself does not produce any conflicting belief masses (when $(Bel \odot Bel)(\emptyset) = 0$, i.e., $Bel \odot Bel = Bel \oplus Bel$), i.e. whenever $Pl(\omega_i) = 1$ for some $\omega \in \Omega_n$. Otherwise, BF is ***conflicting***, i.e., it contains some internal conflict [9].

Let us recall U_n the ***uniform Bayesian belief function***[1] [9], i.e., the uniform probability distribution on Ω_n, and ***normalized plausibility of singletons***[2] of Bel: the BBF (probability distribution) $Pl\_P(Bel)$ such, that $(Pl\_P(Bel))(\omega_i) = \frac{Pl(\{\omega_i\})}{\sum_{\omega\in\Omega} Pl(\{\omega\})}$ [3, 7].

Let us define an ***indecisive (indifferent) BF*** as a BF, which does not prefer any $\omega_i \in \Omega_n$, i.e., BF which gives no decisional support for any ω_i, i.e., BF such that $h(Bel) = Bel \oplus U_n = U_n$, i.e., $Pl(\{\omega_i\}) = const.$, i.e., $(Pl\_P(Bel))(\{\omega_i\}) = \frac{1}{n}$.

Belief Functions on 2-Element Frame of Discernment; Dempster's Semigroup

Let us recall **basic algebraic notions** like ***a semigroup*** (an algebraic structure with an associative binary operation), ***a group*** (a structure with an associative binary operation, with a unary operation of inverse, and with a neutral element), ***a neutral element*** n $(n * x = x)$, ***an absorbing element*** a $(a * x = a)$, ***a homomorphism*** f $(f(x * y) = f(x) * f(y))$, etc.

We assume $\Omega_2 = \{\omega_1, \omega_2\}$, in this subsection. There are only three possible focal elements $\{\omega_1\}, \{\omega_2\}, \{\omega_1, \omega_2\}$ and **any normalized basic belief assignment (bba)** m **is defined by a pair** $(a, b) = (m(\{\omega_1\}), m(\{\omega_2\}))$ as $m(\{\omega_1, \omega_2\}) = 1 - a - b$; this is called ***Dempster's pair*** or simply ***d-pair*** in [4, 6, 14, 15] (it is a pair of reals such that $0 \le a, b \le 1, a + b \le 1$).

Extremal d-pairs are the pairs corresponding to BFs for which either $m(\{\omega_1\}) = 1$ or $m(\{\omega_2\}) = 1$, i.e., $(1,0)$ and $(0,1)$. The **set of all non-extremal d-pairs is denoted as** D_0; the set of all non-extremal ***Bayesian d-pairs*** (i.e. d-pairs corresponding to Bayesian BFs, where $a + b = 1$) is denoted as G; the set of d-pairs such that $a = b$ is denoted as S (set of indecisive[3] d-pairs), the set where $b = 0$ as S_1, and analogically, the set where $a = 0$ as S_2 (***simple support BFs***). ***Vacuous BF*** is denoted as $0 = (0,0)$ and there is a special BF (d-pair) $0' = (\frac{1}{2}, \frac{1}{2}) = U_2$, see Figure 1.

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[1] U_n which is idempotent w.r.t. Dempster's rule $\oplus$, and moreover neutral on the set of all BBFs, is denoted as ${}_{nD}0'$ in [7], $0'$ comes from studies by Hájek & Valdés.

[2] Plausibility of singletons is called *contour function* by Shafer in [17], thus $Pl\_P(Bel)$ is a normalization of contour function in fact.

[3] BFs (a, a) from S are called *indifferent* BFs by Haenni [13].

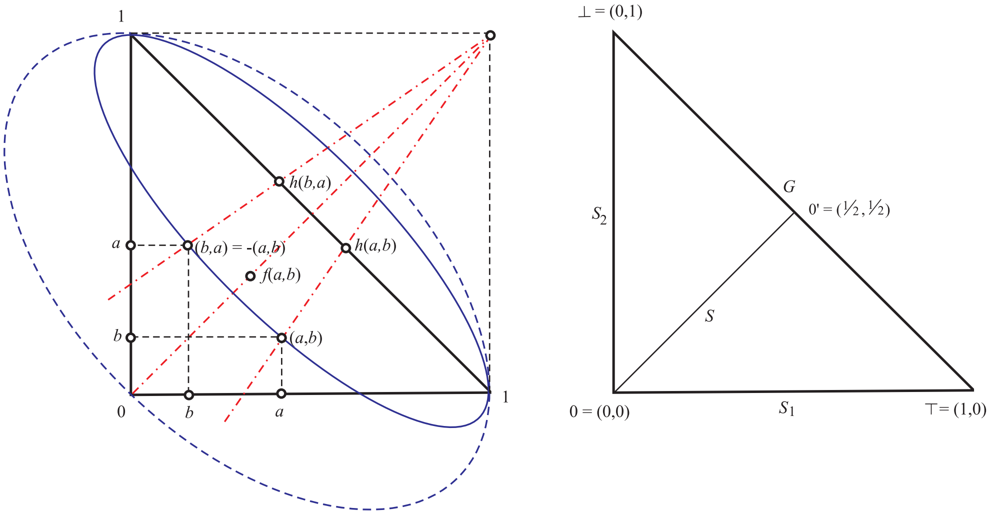

Figure 1: Dempster's semigroup D_0. Homomorphism h is in this representation a projection to group G along the straight lines running through the point $(1,1)$. All the Dempster's pairs lying on the same ellipse (running through the points $(0,1)$ and $(1,0)$) are mapped by homomorphism f to the same d-pair in semigroup S.

The ***(conjunctive) Dempster's semigroup*** $\mathbf{D}_0 = (D_0, \oplus, 0, 0')$ is the set D_0 endowed with the binary operation $\oplus$ (i.e. with the Dempster's rule) and two distinguished elements 0 and $0'$. Dempster's rule can be expressed by the formula $(a,b) \oplus (c,d) = (1 - \frac{(1-a)(1-c)}{1-(ad+bc)}, 1 - \frac{(1-b)(1-d)}{1-(ad+bc)})$ for d-pairs [14]. In D_0 it is defined further: $-(a,b) = (b,a)$, $h(a,b) = (a,b) \oplus 0' = (\frac{1-b}{2-a-b}, \frac{1-a}{2-a-b})$, $h_1(a,b) = \frac{1-b}{2-a-b}$, $f(a,b) = (a,b) \oplus (b,a) = (\frac{a+b-a^2-b^2-ab}{1-a^2-b^2}, \frac{a+b-a^2-b^2-ab}{1-a^2-b^2})$; $(a,b) \leq (c,d)$ iff $[h_1(a,b) < h_1(c,d)$ or $h_1(a,b) = h_1(c,d)$ and $a \leq c]$ [4].

The principal properties of $\mathbf{D}_0$ are summarized by the following theorem:

Theorem 1 *(i) The* **Dempster's semigroup** $\mathbf{D}_0$ *with the relation* $\leq$ *is an ordered commutative (Abelian) semigroup with the neutral element* 0*;* $0'$ *is the only non-zero idempotent of* $\mathbf{D}_0$.
(ii) $\mathbf{G} = (G, \oplus, -, 0', \leq)$ **is an ordered Abelian group**, *isomorphic to the additive group of reals with the usual ordering. Let us denote its negative and positive cones as* $G^{\leq 0'}$ *and* $G^{\geq 0'}$.
(iii) The sets S, S_1, S_2 *with the operation* $\oplus$ *and the ordering* $\leq$ *form* **ordered commutative semigroups** *with neutral element* 0*; they are all isomorphic to the positive cone of the additive group of reals.*
(iv) h **is ordered homomorphism***:* $(D_0, \oplus, -, 0, 0', \leq) \longrightarrow (G, \oplus, -, 0', \leq)$*;* $h(Bel) = Bel \oplus 0' = Pl\_P(Bel)$*, i.e., the normalized plausibility probabilistic transformation.*
(v) f **is homomorphism***:* $(D_0, \oplus, -, 0, 0') \longrightarrow (S, \oplus, -, 0)$*; (but, not an ordered one).*

For proofs see [14, 15, 20]. Let us denote $h^{-1}(x) = \{w \,|\, h(w) = x\}$ and similarly $f^{-1}(x) = \{w \,|\, f(w) = x\}$. Using the theorem, see (iv) and (v), we can express $\oplus$ as:

$$(x \oplus y) = h^{-1}(h(x) \oplus h(y)) \cap f^{-1}(f(x) \oplus f(y)).$$

Let us denote $D_0^{\geq 0} = \{(a,b) \in D_0 \,|\, (a,b) \geq 0\}$ and analogically $D_0^{\leq 0'} = \{(a,b) \leq 0'\}$.

BFs on n-Element Frames of Discernment

Analogically to the case of Ω_2, **we can represent a BF** on any n-element frame of discernment Ω_n by an enumeration of its m values (bbms), i.e., **by a** (2^n-2)**-tuple** $(a_1, a_2, ..., a_{2^n-2})$, or as a (2^n-1)-tuple $(a_1, a_2, ..., a_{2^n-2}; a_{2^n-1})$ when we want to explicitly mention also the redundant value $m(\Omega) = a_{2^n-1} = 1 - \sum_{i=1}^{2^n-2} a_i$. For BFs on Ω_3 we use $(a_1, a_2, ..., a_6; a_7) = (m(\{\omega_1\}), m(\{\omega_2\}), m(\{\omega_3\}), m(\{\omega_1, \omega_2\}), m(\{\omega_1, \omega_3\}), m(\{\omega_2, \omega_3\}); m(\{\Omega_3\}))$.

Unfortunately, no algebraic analysis of BFs on Ω_n for $n > 2$ has been presented till now.

[4]Note, that $h(a,b)$ is an abbreviation for $h((a,b))$, similarly for $h_1(a,b)$ and $f(a,b)$.

3 Non-conflicting and Conflicting Parts of Belief Functions on 2-Element Frames of Discernment $\Omega_2 = \{\omega_1, \omega_2\}$

Proposition 1 **BF *Bel* on Ω_2 is non-conflicting iff $Bel \in S_1 \cup S_2$.**

We will use the important property of Dempster's sum, which is respecting the homomorphisms h and f, i.e., respecting the **h-lines** and **f-ellipses**, when BFs are combined on two-element frame Ω_2.

Proposition 2 **Any belief function $(a, b) \in \Omega_2$ is the result of Dempster's combination of BF $(a_0, b_0) \in S_1 \cup S_2$ and a BF $(s, s) \in S$, such that (a_0, b_0) has the same plausibility decision support (same normalized plausibility) for the elements of Ω_2 as (a, b) does.** *(Trivially, $(s,s) = (0,0) \oplus (s,s)$ for $(s,s) \in S$, and $(a_0, b_0) = (a_0, b_0) \oplus (0,0)$ for elements of S_1 and S_2).* **$(a_0, b_0) \in S_1 \cup S_2$ has no internal conflict, and (s, s) does not prefer any of the elements of Ω_2. Let us call (a_0, b_0) a *non-conflicting part* of (a, b). There is $(a_0, b_0) = (\frac{a-b}{1-b}, 0)$ for $a \geq b$ and $(a_0, b_0) = (0, \frac{b-a}{1-a})$ for $a \leq b$.**

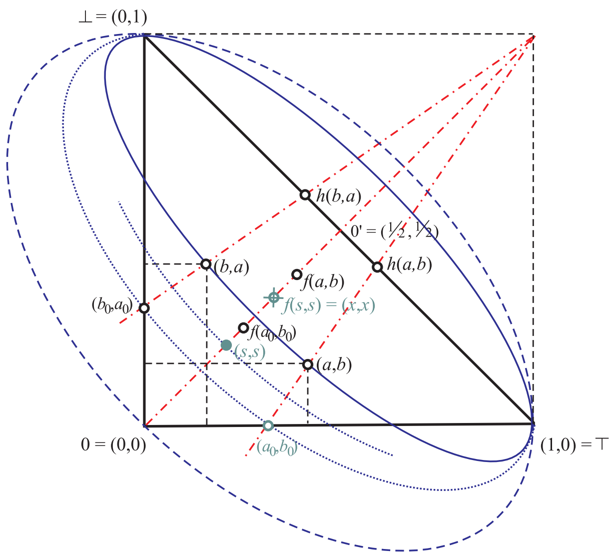

Figure 2: Conflicting and non-conflicting parts of BF on 2-element frame of discernment.

Let us look for (s, s) from the proposition now. It holds true that $(a, b) = (a_0, b_0) \oplus (s, s)$, thus it also holds true $f(a, b) = f(a_0, b_0) \oplus f(s, s)$. We can compute $f(s, s)$ from it, see Lemma 1 (i). Its existence and also a possibility of its computation follows the fact, that S is isomorphic to the positive cone of the additive group of reals, or a property subtraction in S as a substructure of algebraic structure dempsteroid [14, 15].

We already have $f(s, s)$, the rest is computation of (s, s) as S-preimage of $f(s, s)$, see Lemma 1(ii). Finally, we obtain a summarization in Theorem 2.

Lemma 1 *(i) For any BFs (u, u), (v, v) on S, such that $u \leq v$, we can compute their* **Dempster's 'difference'** *(x, x) such that $(u, u) \oplus (x, x) = (v, v)$, where $(x, x) = (\frac{v-u}{1-3u+2uv}, \frac{v-u}{1-3u+2uv})$.*
(ii) For any BF (w, w) on S, we can compute its **Dempster's 'half'** *(s, s) such that $(s, s) \oplus (s, s) = (w, w)$, where $(s, s) = (\frac{1-\sqrt{1-3w+2w^2}}{3-2w}, \frac{1-\sqrt{1-3w+2w^2}}{3-2w}) = (\frac{1-\sqrt{(1-w)(1-2w)}}{3-2w}, \frac{1-\sqrt{(1-w)(1-2w)}}{3-2w})$.*
(iii) There is no Dempster's 'difference' on D_0 in general.

For derivation and verification of formulas and also for two alternative proofs of the theorem see proceedings.

Theorem 2 *Any BF (a, b) on 2-element frame of discernment Ω_2 is* **Dempster's sum** *of its* **unique *non-conflicting part*** *$(a_0, b_0) \in S_1 \cup S_2$ and of its* **unique *conflicting part*** *$(s, s) \in S$, which does not prefer any element of Ω_2, i.e. $(a, b) = (a_0, b_0) \oplus (s, s)$. It holds true that $s = \frac{b(1-a)}{1-2a+b-ab+a^2} = \frac{b(1-b)}{1-a+ab-b^2}$ and $(a, b) = (\frac{a-b}{1-b}, 0) \oplus (s, s)$ for $a \geq b$; and similarly that $s = \frac{a(1-b)}{1+a-2b-ab+b^2} = \frac{a(1-a)}{1-b+ab-a^2}$ and $(a, b) = (0, \frac{b-a}{1-a}) \oplus (s, s)$ for $a \leq b$.*

We can summarize formulas from Theorem 2 as $(a, b) = (a_0, b_0) \oplus (s, s) = (max(\frac{a-b}{1-b}, 0), max(\frac{b-a}{1-a}, 0)) \oplus (\frac{min(a,b)(1-min(a,b))}{1+ab-max(a,b)-min^2(a,b)}, \frac{min(a,b)(1-min(a,b))}{1+ab-max(a,b)-min^2(a,b)})$. And analogically for the second expression of s [10].

4 Non-conflicting Part of BFs on General Finite Frames of Discernment $\Omega_n = \{\omega_1, ..., \omega_n\}$

Let us start with a characterization of the set of non-conflicting BFs.

Proposition 3 **The set of non-conflicting BFs is just the set of all BFs such, that all focal elements of a BF have non-empty intersection** *(i.e.* **consistent BFs** *in Cuzzolin's terminology). Consonant BFs are a special case of non-conflicting BFs.*

We would like to verify that Theorem 2 holds true also for BFs defined on general finite frames, i.e., to verify the following hypothesis:

Hypothesis 1 *We can represent* **any BF** *Bel on n-element frame of discernment Ω_n as* **Dempster's sum** *$Bel = Bel_0 \oplus Bel_S$ of* **non-conflicting BF** *Bel_0 and of* **indecisive conflicting BF** *Bel_S which has no decisional support, i.e. which does not prefer any element of Ω_n to the others, see Fig. 3.*

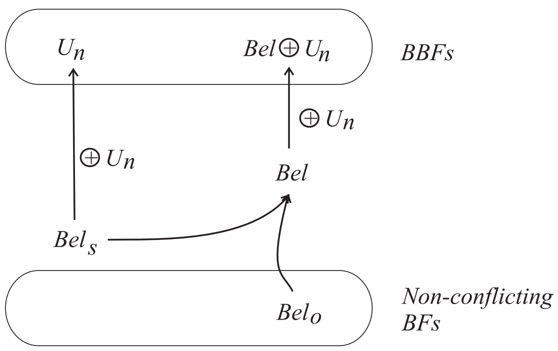

Figure 3: Schema of Hypothesis 1.

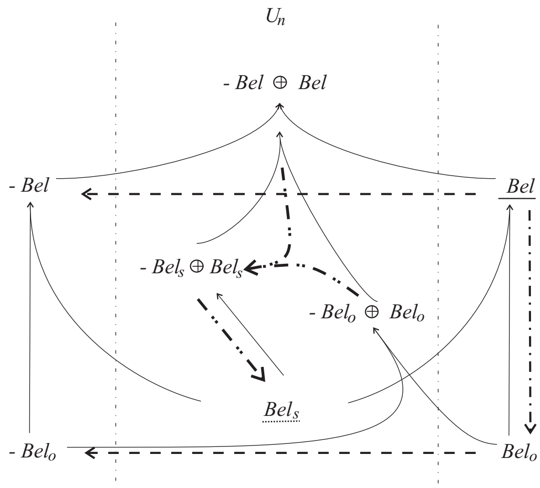

Figure 4: Schema of a decomposition of BF Bel.

We would like to follow the idea from the case of two-element frames, see Figure 4. **Unfortunately, there was not presented any algebraic description of BFs defined on n-element frames till now.** We have nothing like Dempster's semigroup for n-element frames, we have no n-versions of $-Bel$ and of homomorphisms f and h, neither group properties of a set of indecisive BFs.

An issue of homomorphism h is quite promising:

Theorem 3 *The mapping $h(Bel) = Bel \oplus U_n = Pl\_P(Bel)$ is an* **homomorphism** *of an algebra of BFs on an n-element frame of discernment with the binary operation of Dempster's sum $\oplus$ and two nulary operations (constants) 0 and U_n to the algebra of BBFs on Ω_n with binary operation $\oplus$ and nulary operation U_n.*

A procedure for computing of unique consonant BF Bel_0 to any $h(Bel)$:
Let there are k different values $h(Bel)(\omega_i) = h_i(Bel)$ for $i = 1, ..., n$, thus $1 \leq k \leq n$. According to this, we have splitting of the frame Ω into k disjoint subsets $\Omega = \Omega_1 \cup \Omega_2 \cup ... \cup \Omega_k$, such the the elements of the same subset have the same value $h(Bel)(\omega)$. Let $\Omega_1 = \{\omega_{11}, ..., \omega_{1j_1}\}$ be a set of elements of the frame with the highest m-value (bbm) $(h(Bel)(\omega_{11}) = h(Bel)(\omega_{12}) = ... = h(Bel)(\omega_{1j_1})$, where $1 \leq j_1 \leq n-k+1)$, and $\Omega_2 = \{\omega_{21}, ..., \omega_{2j_2}\}$ be a set of elements with the 2nd highest bbm $(h(Bel)(\omega_{21})$; $1 \leq j_2 \leq n - j_1 - k + 2)$, then we define $m_w(\Omega_1) = h(Bel)(\omega_{11}) - h(Bel)(\omega_{21})$, further we define $m_w(\Omega_1 \cup \Omega_2) = h(Bel)(\omega_{21}) - h(Bel)(\omega_{31})$, where $h(Bel)(\omega_{31})$ is the 3rd largest m-value of $h(Bel)$. We continue similarly defining $m_w(\bigcup_{i=1}^{m} \Omega_i) = h(Bel)(\omega_{m1}) - h(Bel)(\omega_{(m+1)1})$, where $\Omega_i = \{\omega_{i1}, ..., \omega_{ij_i}\}$ is the set of elements with the i-th highest m-value of $h(Bel)$, until $m_w(\Omega) = h(Bel)(\omega_{k1})$ is defined, where $\Omega_k = \{\omega_{k1}, ..., \omega_{kj_k}\}$ is the set of elements with the least (possibly zero), m-value $h(Bel)(\omega_{k1})$, $j_k = n - \sum_{i=1}^{k-1} j_i$. $m_w(\bigcup_{i=1}^{m} \Omega_i) > 0$ for all $m < k$ because less value is always decreased, $m_w(\Omega_k) \geq 0$, $\sum_{m=1}^{k} m_w(\bigcup_{i=1}^{m} \Omega_i) = h(Bel)(\omega_{11})$. Then m_0 is a normalization of working bba m_w, thus focal elements of m_0 are nested, $Pl(\omega) = 1$ for $\omega \in \Omega_1$, hence Bel_0 is normalized consonant, i.e., non-conflicting BF.

Finally, we can simplify the construction of Bel_0 in the following way: there is one normalization in computation of $Bel \oplus U_n = Pl\_P(Bel)$ and the following normalization in the transformation of m_w to m_0. Normalization commutes with the construction of m_w from $Pl\_P(Bel)$, thus when computing Bel_0, we can use $Pl(Bel)$ instead of $h(Bel) = Pl\_P(Bel)$ and apply only one normalization in the end, where normalization factor is the multiple of the original ones. Thus we obtain $m'_w(\{\omega_{11}, ..., \omega_{1j_1}\}) = Pl(Bel)(\omega_{11}) - Pl(Bel)(\omega_{21})$, etc. **This computational simplification** is important also from the theoretical point of view, because it **removes Dempster's rule $\oplus$ hidden in h** from the construction of Bel_0. Hence **any Bel has defined its non-conflicting part Bel_0 independently of any belief combination rule**.

Lemma 2 ***For any BF*** Bel ***defined on*** Ω_n ***there exists unique consonant BF*** Bel_0 ***such that,*** $h(Bel_0 \oplus Bel_S) = h(Bel)$ ***for any BF*** Bel_S ***such that*** $Bel_S \oplus U_n = U_n$***.***

Let us notice, that the stronger statement for a general non-conflicting BFs does not hold true on Ω_n. There could be several different non-conflicting BFs Bel_i such that $h(Bel_i \oplus Bel_S) = h(Bel)$ for any indecisive BF B_S. See, the following example.

Example 1 *To BF* $Bel = (0.25, 0.175, 0.075, 0.35, 0.15, 0)$ *with* $h(Bel) = (0.25, 0.175, 0.075, 0.35, 0.15, 0) \oplus (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0, 0, 0) = (0.50, 0.35, 0.15, 0, 0, 0)$ *there are following non-conflicting BFs:* $Bel_0 = (0.3, 0, 0, 0.4, 0, 0; 0.3)$*,* $Bel_1 = (0, 0, 0, 0.7, 0.3, 0; 0)$*,* $Bel_2 = (0.2, 0, 0, 0.5, 0.1, 0; 0.2)$*;* $Pl_i(\{\omega_1\}) = 1$*, thus* Bel_i*s are all non-conflicting, we can simply verify that* $h(Bel_i) = h(Bel)$*, thus* $(Bel_i \oplus Bel_S) \oplus U_3 = Bel_i \oplus (Bel_S \oplus U_3) = Bel_i \oplus U_3 = h(Bel)$*.*

Let us turn our attention to $f(Bel)$ and $-Bel$ now. $f(a, b) = -(a, b) = (b, a)$ on Ω_2, thus we will try to generalize $-Bel$ to BFs on Ω_n now. We have nothing like S defined for BFs on Ω_n, thus we suppose $h(Bel \oplus -Bel) = U_n$ for $-Bel$.

An idea of complements. On Ω_2 it holds true that $-m(\{\omega_1\}) = m(\{\omega_2\}) = m(\Omega_2 \setminus \{\omega_1\})$, $-m(\{\omega_2\}) = m(\Omega_2 \setminus \{\omega_2\})$, and $-m(\Omega_2) = m(\Omega_2)$[5]. Unfortunately, the simple idea to define $-m$ as $-m(X) = m(\Omega_n \setminus X)$ does not work in general, not even for general consonant BFs, e.g., for $Bel = (0.5, 0, 0, 0.2, 0, 0; 0.3)$ and $\sim Bel = (0, 0, 0.2, 0, 0, 0.5; 0.3)$ we have $Bel \oplus \sim Bel = (\frac{15}{61}, \frac{10}{61}, \frac{6}{61}, \frac{6}{61}, \frac{0}{61}, \frac{15}{61}; \frac{9}{61})$, $(\frac{15}{61}, \frac{10}{61}, \frac{6}{61}, \frac{6}{61}, \frac{0}{61}, \frac{15}{61}; \frac{9}{61}) \oplus (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0, 0, 0; 0) = (\frac{30}{70}, \frac{40}{70}, \frac{30}{70}, 0, 0, 0; 0) = (\frac{3}{7}, \frac{4}{7}, \frac{3}{7}, 0, 0, 0; 0) \neq U_3$. Thus $h(Bel \oplus \sim Bel) \neq U_n$, hence $\sim Bel \neq -Bel$. The idea of complements $(\Omega \setminus X)$ works only in some special cases, e.g., for $(0.7, 0, 0, 0, 0, 0) \oplus (0, 0, 0, 0, 0, 0.7) = (21/51, 0, 0, 0, 0, \frac{21}{51}) \doteq (0.41, 0, 0, 0, 0, 0.41)$, $h(0.41, 0, 0, 0, 0, 0.41) = U_3$ on Ω_3 and for other simple support BFs in general.

To **simplify the investigated situation**, we will start with **qBBFs on 3-element frame** of discernment Ω_3 (i.e., with BFs such that $m(X) = 0$ for $|X| = 2$). The set of qBBFs on Ω_3 can be represented by a three dimensional triangle which simply generalizes the triangle of Dempster's pairs, see Figure 5. Unfortunately, the only consonant, i.e. non-conflicting, BFs are singleton simple support functions as $(a, 0, 0, 0, 0, 0; 1-a)$, thus only a small part of the triangle is mapped to non-conflicting BFs within the triangle (Bel_0 is outside of the triangle for a majority of qBBFs). Thus, this is not a good domain to search for $-Bel_0$.

Let us look at BBFs now, i.e. BFs as $(a, b, c, 0, 0, 0; 0) = (a, b, 1-a-b, 0, 0, 0; 0)$. Let $-(a, b, 1-a-b, 0, 0, 0) = (x, y, 1-x-y, 0, 0, 0)$, thus $-(a, b, 1-a-b, 0, 0, 0) \oplus (x, y, 1-x-y, 0, 0, 0) = U_3$ should hold true. From it we obtain $x = \frac{bc}{ab+ac+bc}$, $y = \frac{ac}{ab+ac+bc}$ and $1 - x - y = z = 1 - \frac{bc+ac}{ab+ac+bc} = \frac{ab}{ab+ac+bc}$. E.g. $-(0.5, 0.3, 0.2, 0, 0, 0) = (\frac{6}{31}, \frac{10}{31}, \frac{15}{31}, 0, 0, 0)$, as $x = \frac{0.3 \cdot 0.2}{0.5 \cdot 0.3 + 0.5 \cdot 0.2 + 0.3 \cdot 0.2}$, $y = \frac{5 \cdot 2}{5 \cdot 3 + 5 \cdot 2 + 3 \cdot 2}$ $z = \frac{5 \cdot 3}{5 \cdot 3 + 5 \cdot 2 + 3 \cdot 2}$. Thus we have $-Bel$ for any BBF $(a, b, 1-a-b, 0, 0, 0)$ on Ω_3 such that $0 < a, b < 1, a + b < 1$.

Analogically we can generalize the $-Bel$ to BBFs on Ω_n, we obtain $x_1 = 1/(1 + \sum_{i=2}^{n} \frac{a_1}{a_i})$, $x_i = \frac{a_1}{a_i} x_1$, or similarly to x_1: $x_i = 1/(1 + \sum_{i \neq j} \frac{a_i}{a_j})$. An alternative expression for x_i is $x_i = \frac{\prod_{i \neq j} a_j}{\sum_{k=1}^{n} \prod_{j \neq k} a_j}$.

Lemma 3 *For* **any BBF** $(a_1, a_2, ..., a_n, 0, 0, ..., 0; 0)$ *such that,* $a_i > 0$ *for* $i = 1, ..., n$*, there exists* **uniquely defined** $-(a_1, a_2, ..., a_n, 0, 0, ..., 0; 0) = (x_1, x_2, ..., x_n, 0, 0, ..., 0; 0) = (1/(1 + \sum_{i=2}^{n} \frac{a_1}{a_i})$, $\frac{a_1}{a_2} x_1, \frac{a_1}{a_3} x_1, ..., \frac{a_1}{a_n} x_1, 0, 0, ..., 0; 0)$ *such that,*

$$(a_1, a_2, ..., a_n, 0, 0, ..., 0) \oplus -(a_1, a_2, ..., a_n, 0, 0, ..., 0) = U_n.$$

[5] Note that $-m(X)$ is an abbreviation for $(-m)(X)$, thus both $m(X)$ and $-m(X)$ may be positive in general. Specially $-m(\Omega_2)$ is an abbreviation for $(-m)(\Omega_2)$, thus $-m(\Omega_2) = m(\Omega_2)$, where both sides of the equation are positive in general.

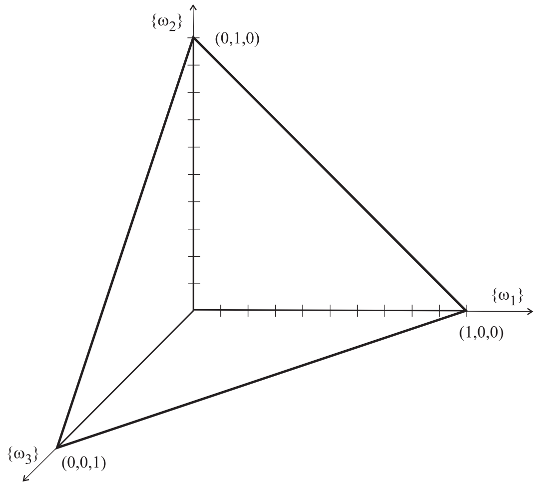

Figure 5: Quasi Bayesian BFs on 3-el. Ω_3.

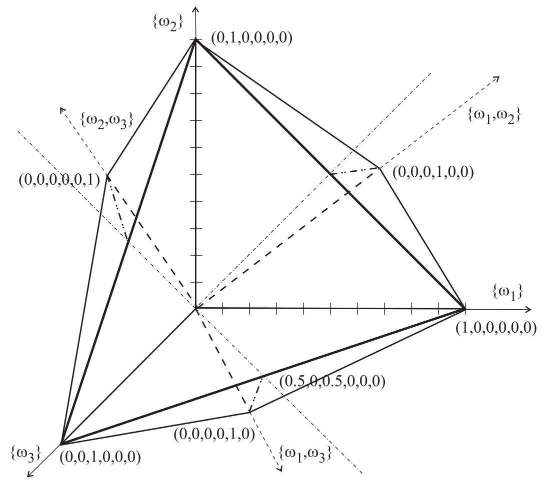

Figure 6: General BF on 3-element frame Ω_3.

We have $-Bel$ for a simple support function (SSF), thus also for simple support BBFs (i.e. categorical BBFs), see e.g., $-(1,0,0,0,0,0) = (0,0,0,0,0,1)$; but a definition of $-Bel$ for BBFs like $(a, 1-a, 0, 0, ..., 0)$ remains still open for more-element frames Ω_n, $n > 2$.

Summarising the previous results, we can step by step compute $h(Bel)$, $-h(Bel)$ and $(-h(Bel))_0$ from any Bel such that $Pl(\{\omega_i\}) > 0$ for all $\omega_i \in \Omega_n$, see Fig. 7. Thus the following holds true:

Theorem 4 *For* **any BF** *Bel defined on Ω_n there* **exists unique consonant BF** *Bel_0 such that,*

$$h(Bel_0 \oplus Bel_S) = h(Bel)$$

for any BF Bel_S such that $Bel_S \oplus U_n = U_n$. If for $h(Bel) = (h_1, h_2, ..., h_n, 0, 0, ..., 0)$ holds true that, $0 < h_i < 1$, then further exists unique BF $-Bel_0$ such that,

$$h(-Bel_0 \oplus Bel_S) = -h(Bel) \text{ and } h(Bel_0) \oplus -h(Bel_0) = U_n.$$

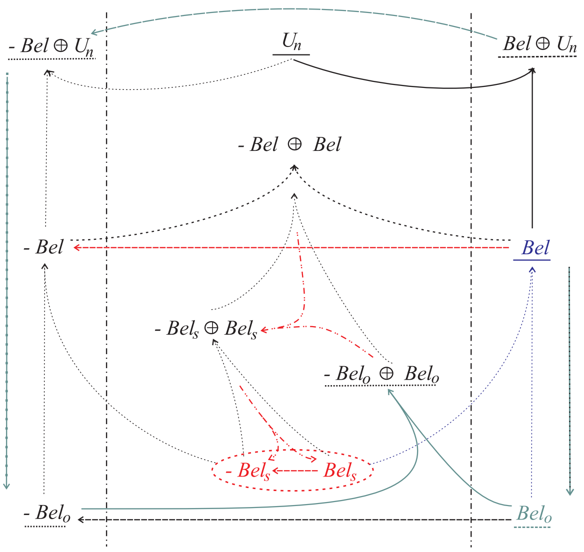

Figure 7: Detailed schema of a decomposition of BF Bel.

Corollary 1 *(i) For any consonant BF Bel such that $Pl(\{\omega_i\}) > 0$ there exist a unique BF $-Bel$; $-Bel$ is consonant in this case.*
(ii) There is one-to-one correspondence between Bayesian BFs and consonant BFs.

We observed that $-m(X) = m(\Omega \setminus X)$ for $X \subset \Omega$ and SSF m. In proceedings, it is verified that the definition of $-Bel$ using $-h(Bel)$ agree with this observation.
For completion of the diagram in Figure 7, we need a definition of $-Bel$ for general BFs on Ω to compute $Bel \oplus -Bel$ and analysis of indecisive BFs (i.e. BFs Bel such that, $h(Bel) = U_n$) to compute $Bel_S \oplus -Bel_S$ and resulting Bel_S and specify conditions under which Bel_S is defined and unique. Hence **an algebraic analysis of BFs on a general finite frame of discernment is required.**

6 Ideas for Future Research

- Algebraic analysis of BFs on a 3-element frame Ω_3.
- Algebraic analysis of BFs on a general finite frame Ω_n.
- Existence and uniqueness of a conflitcting part of a general BF on a general finite frame.
- Interpretation of (s, s) on Ω_2 and of a conflicting part of a general BF.

5 Comments on other belief combination rules and probabilistic transformations

Other belief combination rules

As it was already mentioned, the non-conflicting part Bel_0 of a belief function Bel defined above is independent from Dempster's rule of combination, as we can use the representation of homomorphism h using normalized plausibility of singletons $Pl\_P(Bel)$ instead of the original $h(Bel) = Bel \oplus U_n$. Thus Bel_0 can be computed without any relation to Dempster's rule and $Pl\_P(Bel_0) = Pl\_P(Bel)$ **independently from any combination rule**.

On the other hand $Pl\_P(Bel) \neq Bel_0 \textcircled{Y} U_n$, $Pl\_P(Bel) \neq Bel_0 \textcircled{DP} U_n$, $Pl\_P(Bel) \neq Bel_0 \textcircled{\cup} U_n$, see Example 2. Even $Pl\_P(Bel) \neq Pl\_P(Bel_0 \odot U_n)$, **where $\odot$ is either $\textcircled{Y}$, $\textcircled{DP}$, $\textcircled{\cup}$ or** some other combination rule. The equality holds true only for Dempster's rule: $Pl\_P(Bel) = Bel_0 \oplus U_n$; in the case of un-normalized conjunctive rule $\textcircled{\cap}$ we can apply additional normalization to obtain the equality, thus we have normalized conjuntive rule, i.e., Dempster's rule $\oplus$ again. (? Pl_P in TBM? ... $Pl\_P(Bel_0 \textcircled{\cap} U_n) = BetP(Bel_0 \textcircled{\cap} U_n)$?)

Example 2 *Let us take* $Bel = (0.3, 0.2, 0.1, 0.2, 0.1, 0.0; 0.1)$*, thus there is* $Pl = (0.7, 0.5, 0.3, 0.9, 0.8, 0.7; 1.0)$, $Pl\_P(Bel) = (\frac{7}{15}, \frac{5}{15}, \frac{3}{15})$*, and* $Bel_0 = (\frac{2}{7}, 0, 0, \frac{2}{7}, 0, 0; \frac{3}{7})$*. Hence we obtain*
$(\frac{2}{7}, 0, 0, \frac{2}{7}, 0, 0; \frac{3}{7}) \oplus (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0, 0, 0; 0) = (\frac{7}{15}, \frac{5}{15}, \frac{3}{15}, 0, 0, 0; 0)$*; but*
$(\frac{2}{7}, 0, 0, \frac{2}{7}, 0, 0; \frac{3}{7}) \textcircled{Y} (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0, 0, 0; 0) = (\frac{7}{21}, \frac{5}{21}, \frac{3}{21}, 0, 0, 0; \frac{6}{21}) \neq (\frac{7}{15}, \frac{5}{15}, \frac{3}{15}, 0, 0, 0; 0)$,
$(\frac{2}{7}, 0, 0, \frac{2}{7}, 0, 0; \frac{3}{7}) \textcircled{DP} (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0, 0, 0; 0) = (\frac{7}{21}, \frac{5}{21}, \frac{3}{21}, \frac{2}{21}, \frac{2}{21}, 0; \frac{2}{21}) \neq (\frac{7}{21}, \frac{5}{21}, \frac{3}{15}, 0, 0, 0; 0)$*, and similarly*
$(\frac{2}{7}, 0, 0, \frac{2}{7}, 0, 0; \frac{3}{7}) \textcircled{\cup} (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0, 0, 0; 0) = (\frac{2}{21}, 0, 0, \frac{6}{21}, \frac{2}{21}, 0; \frac{11}{21}) \neq (\frac{7}{15}, \frac{5}{15}, \frac{3}{15}, 0, 0, 0; 0)$*. Further*
$Pl\_P(\frac{7}{21}, \frac{5}{21}, \frac{3}{21}, 0, 0, 0; \frac{6}{21}) = (\frac{13}{33}, \frac{11}{33}, \frac{9}{33}) \neq (\frac{7}{15}, \frac{5}{15}, \frac{3}{15})$,
$Pl\_P(\frac{7}{21}, \frac{5}{21}, \frac{3}{15}, \frac{2}{21}, \frac{2}{21}, 0; \frac{2}{21}) = (\frac{13}{29}, \frac{9}{29}, \frac{7}{29}) \neq (\frac{7}{15}, \frac{5}{15}, \frac{3}{15})$*, and*
$Pl\_P((\frac{2}{21}, 0, 0, \frac{6}{21}, \frac{2}{21}, 0; \frac{11}{21}) = (\frac{21}{51}, \frac{17}{51}, \frac{13}{51}) \neq (\frac{7}{15}, \frac{5}{15}, \frac{3}{15})$.

Nevertheless, if there is a couple of homomorphisms for any combination rule $\odot$ analogous to the couple of morphisms f and h from Dempster's semigroup, then there exists an analogy of Bel_0 also for the combination rule $\odot$.

Other probabilistic transfromations

When expressing h using $Pl\_P(Bel)$ there arises another interesting question about similar kind of non-conflicting part and decomposition of belief functions using a different probabilistic transformations, see Figure 8.

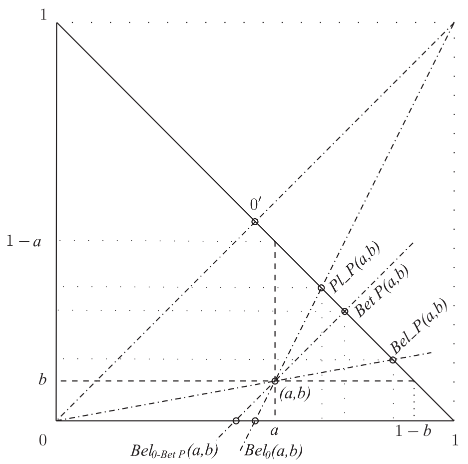

Figure 8: Probabilistic transformations.

Considering Smets' pignistic transformation $BetT$ for computing pignistic probability $BetP$ **we obtain non-conflicting BF** Bel_{0-BetP}, where $m_{w-BetP}(\bigcup_{i=1}^{m} \Omega_i) = |\bigcup_{i=1}^{m} \Omega_i|(h(Bel)(\omega_{m1}) - h(Bel)(\omega_{(m+1)1}))$, which is normalized, hence $m_{0-BetP} = m_{w-BetP}$. $BetT$ does not commute either with Dempster's rule nor with other rules defined for belief combination, thus **we cannot use** Bel_{0-BetP} **for decomposition** of belief functions to conflicting and non-conflicting parts. For counter-examples see [10].

The most perspective probabilistic transformation **is normalized belief of singletons** Bel_P which is compatible with disjunctive rule of combination [7], unfortunately, the reverse transformation maps any Bel and $Bel\_P(Bel)$ to the vacuous belief function $0 = (0, 0, ..., 0; 1)$, which is really non-conflicting, but **it does not reprezent non-conflicting part** of the belief function Bel. In this case **it represents zero conflicting part**, as the disjunctive rule is completely non-conflicting; thus it holds true $Bel = Bel \textcircled{\cup} 0$, where Bel is trivial 'disjunctive non-conflicting' part of itself and 0 is trivial 'disjunctive conflicting' part of any BF Bel.

Moreover, it is possible to show that there is no similar decomposition of belief functions for $\textcircled{Y}$, $\textcircled{DP}$, $\textcircled{\cup}$ and a for a series of other combination rules, see [10]. Any Bayesian BF serves as counter-example there.

7 Conclusion

- **Decomposition** of a belief function (BF) defined on a two-element frame of discernment to Dempster's sum of its **unique non-conflicting** and **unique indecisive conflicting part** is defined and presented here.

- **Homomorphic properties** of mapping $h(Bel) = Bel \oplus U_n$ which corresponds to normalized plausibility of singletons were verified for BFs defined on a general finite frame of discernment. $-Bel$ was generalized to **Bayesian BFs** and for **consonant BFs** on a general n-element frame.

- **Unique consonant non-conflicting part** Bel_0 of a general BF Bel on a finite frame was defined. For specification of a corresponding conflicting part of Bel and its uniqueness/existence properties, **an algebraic analysis of BFs on a general finite frame of discernment is required.**

- Discussion of the topic from the point of view of alternative rules of combination and alternative probabilistic transformations.

- The presented results improve general understanding of belief functions and their combination, especially in conflicting cases.
  They can be used as one of corner-stones to further study of conflicts between belief functions.

Current Related Research

F. Cuzzolin — Consistent transformations of belief functions. ECSQARU 2011 [22]
F. Cuzzolin — Consonant transformations of belief functions. ISIPTA 2011 [23]
Lefevre-Elouedi-Mercier — Partial normalization of conflicting mass $m(\emptyset)$ in TBM. ECSQARU 2011 [24]

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