Robustness of Natural Extension

Matthias C. M. Troffaes[1] Robert Hable[2]

[1]Durham University, United Kingdom

[2]Universität Bayreuth, Germany

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Introduction

Aim

how sensitive is the natural extension of an upper prevision against small perturbations in the assessments?

- various forms and representations of natural extension: does this affect stability?
- necessary and sufficient conditions?
- transform instable problems into stable ones?
- extend earlier work ([1, pp. 118–125, Sec. 5.2] and [2, Sec. 2])

Example

- adapted from Robinson [3, p. 443]
- $\Omega = \{a, b, c, d\}$

$$\overline{P}(I_a + 2I_b/3 + 2I_d) = 1/2 \qquad \overline{P}(I_b + 3I_c) = 3/2$$

- $\overline{E}$ of $2I_b + 2I_c$:

$$\text{maximize} \quad \begin{bmatrix} 0 & 2 & 2 & 0 \end{bmatrix} \begin{bmatrix} x_a \\ x_b \\ x_c \\ x_d \end{bmatrix}$$

subject to

$$x_a \geq 0,\ x_b \geq 0,\ x_c \geq 0,\ x_d \geq 0 \qquad \text{(C)}$$
$$x_a + x_b + x_c + x_d = 1$$

$$\begin{bmatrix} 1 & 2/3 & 0 & 2 \\ 0 & 1 & 3 & 0 \end{bmatrix} \begin{bmatrix} x_a \\ x_b \\ x_c \\ x_d \end{bmatrix} \leq \begin{bmatrix} 1/2 \\ 3/2 \end{bmatrix} \qquad \text{(S)}$$

Instability

- (C) + (S) have a non-empty feasible set

$$\alpha \begin{bmatrix} 1/2 \\ 0 \\ 1/2 \\ 0 \end{bmatrix} + (1-\alpha) \begin{bmatrix} 0 \\ 3/4 \\ 1/4 \\ 0 \end{bmatrix}$$

- so $\overline{E}(2I_b + 2I_c) = 2$
- (C) + (S_ϵ), with

$$\begin{bmatrix} 1 & 2/3-\epsilon & 0 & 2 \\ 0 & 1 & 3 & 0 \end{bmatrix} \begin{bmatrix} x_a \\ x_b \\ x_c \\ x_d \end{bmatrix} \leq \begin{bmatrix} 1/2 \\ 3/2 \end{bmatrix} \qquad (S_\epsilon)$$

has only one feasible solution

$$\begin{bmatrix} 1/2 \\ 0 \\ 1/2 \\ 0 \end{bmatrix}$$

- so, now, $\overline{E}(2I_b + 2I_c) = 1$

arbitrary small perturbation in assessments can have disproportionally large effect on natural extension

Main Results

Canonical Linear Program

$$\text{maximize} \sum_\omega x(\omega) f(\omega) \qquad \text{(1a)}$$
$$\text{subject to } x(\omega) \geq 0,\ \sum_\omega x(\omega) = 1,\ \overline{P}(g) \geq \sum_\omega x(\omega) g(\omega) \qquad \text{(1b)}$$

Canonical Dual Linear Program

$$\text{mimimize } a + \sum_g \lambda_g \overline{P}(g) \qquad \text{(2a)}$$
$$\text{subject to } \lambda_g \geq 0,\ a + \sum_g \lambda_g g \geq f \qquad \text{(2b)}$$

Stability Theorem

Let $\overline{P}$ be any upper prevision. The following conditions are equivalent.

1. The linear program (1) is stable.
2. The linear program (2) is stable.
3. The system of linear inequalities and equalities of (1) is regular.
4. All sufficiently small perturbations of $\overline{P}$ avoid sure loss.
5. There is a linear prevision x such that $\overline{P}(g) > x(g)$ for all g in $\mathcal{K}$.
6. $\overline{P}$ dominates a non-linear linear-vacuous mixture.
7. $\overline{P}$ avoids sure loss and $\underline{E}(g) < \overline{E}(g)$ for all g in $\mathcal{K}$.

Stable Representation Theorem

Let $\overline{P}$ be any upper prevision. The following conditions are equivalent.

1. The linear program (1) has a stable representation.
2. The linear program (2) has a stable representation.
3. The system of linear inequalities and equalities of (1) has a regular representation.
4. There is a linear prevision x in the credal set of $\overline{P}$ such that $x(\omega) > 0$ for all $\omega \in \Omega$.
5. $\overline{P}$ avoids sure loss and $\overline{E}(I_\omega) > 0$ for all $\omega \in \Omega$.

Practical Consequences

for any $0 < \alpha \leq 1$:

- $(1-\alpha)\overline{P} + \alpha \sup_{\omega\in\Omega}$ is *always* stable
- $(1-\alpha)\overline{P} + \alpha \frac{1}{|\Omega|} \sum_{\omega\in\Omega}$ *always* has a stable representation

Open Problems

- extension to non-finite spaces
- conditional lower previsions

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