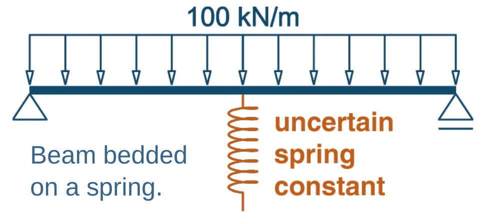


Modelling Uncertainties in Limit State Functions

Thomas Fetz, Universität Innsbruck, Austria

Limit state functions / probability of failure

Limit state functions

$g:\mathcal{X}\subseteq\mathbb{R}^n\to\mathcal{Y}\subseteq\mathbb{R}: x\to y=g(x)$ with basic variables $x=(x_1,\dots,x_n)$ = material properties, loads,...

$y=g(x)\le 0$ means failure of the system.

Failure regions

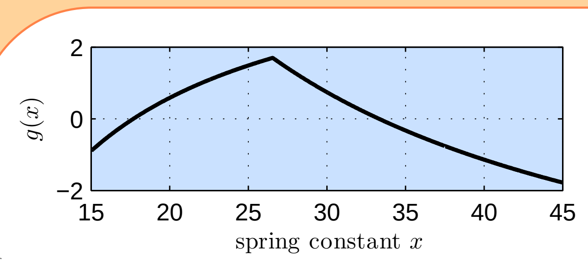
Set $R_f=\{x\in\mathcal{X}:\ g(x)\le 0\}$ described by the indicator function

$$q:\mathcal{X}\to\{0,1\}: x\to\chi(g(x)\le 0).$$

Probability of failure

$$p_f=P(g(X)\le 0)=\int_{\mathcal{X}}\chi(g(x)\le 0)\,f^X(x)\,\mathrm{d}x$$

with density f^X of the random variables X.



Deterministic limit state function g of the beam.

Sets of probability measures $\mathcal{M}_X$ and $\mathcal{M}_Z$

We model the uncertainty of x and z by sets of probability measures. Different notions of independence!

Upper probability of failure / independence

Strong independence: We consider all possible product measures $P_X\otimes P_Z$ for $P_X\in\mathcal{M}_X$ and $P_Z\in\mathcal{M}_Z$:

$$\overline{p}_f^{\mathrm{S}}=\sup\{(P_X\otimes P_Z)(h(x,z)\le 0): P_X\in\mathcal{M}_X,\ P_Z\in\mathcal{M}_Z\}$$

$$=\sup_{\substack{P_X\in\mathcal{M}_X\\P_Z\in\mathcal{M}_Z}}\int_{\mathcal{X}}\int_{\mathcal{Z}}\chi(h(x,z)\le 0)\,\mathrm{d}P_Z(z)\,\mathrm{d}P_X(x)$$

$$=\sup_{\substack{P_X\in\mathcal{M}_X\\q\in Q}}\int_{\mathcal{X}}q(x)\,\mathrm{d}P_X(x)\qquad\text{(set of imprecise failure regions)}$$

$$Q=\left\{q:\ q(x)=\int_{\mathcal{Z}}\chi(h(x,z)\le 0)\,\mathrm{d}P_Z(z),\ P_Z\in\mathcal{M}_Z\right\}$$

- The entire set Q is needed for computations!

Epistemic irrelevance

$$\overline{p}_f^{\mathrm{S}}\le\sup_{P_X\in\mathcal{M}_X}\int_{\mathcal{X}}\sup_{P_Z\in\mathcal{M}_Z}\int_{\mathcal{Z}}\chi(h(x,z)\le 0)\,\mathrm{d}P_Z(z)\,\mathrm{d}P_X(x)$$

$$=\sup_{P_X\in\mathcal{M}_X}\int_{\mathcal{X}}\overline{q}(x)\,\mathrm{d}P_X(x)=:\overline{p}_f^{X\not\to Z}\quad\text{with}$$

$$\overline{q}(x)=\sup_{P_Z\in\mathcal{M}_Z}\int_{\mathcal{Z}}\chi(h(x,z)\le 0)\,\mathrm{d}P_Z(z)=\sup_{q\in Q}q(x).$$

(upper envelope of imprecise failure regions)

- It is sufficient to specify the function $\overline{q}$!
- Each x can choose its own $P_Z\in\mathcal{M}_Z$ or more exactly a $P_Z(\cdot\,|x)$ given x.
- $X\not\to Z$ means that X is epistemically irrelevant to Z or that the basic variables x are epistemically irrelevant to the parameterized limit state functions g_z.
- Epistemic irrelevance is an asymmetric notion of independence meaning (a) but not necessarily (b).

The set $\mathcal{M}_{X\not\to Z}$ of probability measures according to epistemic irrelevance of X to Z is defined by:

$$\mathcal{M}_{X\not\to Z}=\left\{P: P(E)=\int_{\mathcal{X}}\int_{\mathcal{Z}}\chi((x,z)\in E)\,\mathrm{d}P_Z(z|x)\,\mathrm{d}P_X(x)\quad P_X\in\mathcal{M}_X,\ P_Z(\cdot\,|x)\in\mathcal{M}_Z\right\}$$

Random set independence

Joint plausibility measure

$$\mathrm{Pl}(E)=\sum_{i=1}^{|\mathscr{A}|}\sum_{j=1}^{|\mathscr{B}|}m_{\mathscr{A}}(A_i)m_{\mathscr{B}}(B_j)\overbrace{\chi(E\cap(A_i\times B_j)\ne\varnothing)}^{\sup_{P\in\mathcal{M}(A_i\times B_j)}P(E)}$$

In our case: f_a^X and f_b^Z are involved. The "correct" generalization of Pl would be to consider all possible combinations of f_a^X and f_b^Z. $\to$ High computational effort, no independence on the level of f_a^X and f_b^Z.

Here: We replace $\sup_{P\in\mathcal{M}(A_i\times B_j)}P(E)$ by ($\star$) or ($\diamond$).

For Dirac measures instead of f_a^X and f_b^Z all these approaches coincide with Pl.

Parameterized limit state functions

Parameterization g_z of g by $z\in\mathcal{Z}\subseteq\mathbb{R}^m$ using

$$h:\mathcal{X}\times\mathcal{Z}\to\mathcal{Y}:(x,z)\to h(x,z)=g_z(x)$$

where again $y=h(x,z)\le 0$ means failure.

Imprecise failure regions described by

$$q:\mathcal{X}\to[0,1]:x\to\int_{\mathcal{Z}}\chi(h(x,z)\le 0)\,f^Z(z)\,\mathrm{d}z,$$

cf. membership functions of fuzzy sets.

Probability of failure

$$p_f=\int_{\mathcal{X}}\int_{\mathcal{Z}}\chi(h(x,z)\le 0)\,f^Z(z)\,\mathrm{d}z\,f^X(x)\,\mathrm{d}x$$

assuming independence of X and Z.

The function $p_f(a,b)$

$$p_f(a,b)=\int_{\mathcal{X}}\int_{\mathcal{Z}}\chi(h(x,z)\le 0)\,f_b^Z(z)\,\mathrm{d}z\,f_a^X(x)\,\mathrm{d}x$$

a and b are parameters of the densities f_a^X, f_b^Z, e. g. $f_{(\mu,\sigma)}^X$ density of a Gaussian distribution $\mathcal{N}(\mu,\sigma^2)$.

Generating sets $\mathcal{M}_X$ and $\mathcal{M}_Z$

Parameters a and b are assumed to be uncertain.

Uncertainty of a and b modelled by sets A and B

$$\mathcal{M}_X=\left\{P: P(E)=\int_{A}\int_{\mathcal{X}}\chi(x\in E)\,f_a^X(x)\,\mathrm{d}x\,\mathrm{d}P_A(a),\quad P_A\in\mathcal{M}(A)\right\}$$

where $\mathcal{M}(A)=\{P:\ P(A)=1\}$ is the set of all probability measures living on the set $A\subseteq\mathcal{A}$.

Uncertainty of a and b modelled by random sets

Finite random sets $\mathscr{A}$, $\mathscr{B}$ with focal sets A_i, B_j and weights $m_{\mathscr{A}}(A_i)$, $m_{\mathscr{B}}(B_j)$.

$$\mathcal{M}_X=\left\{P:\ P(E)=\sum_{i=1}^{|\mathscr{A}|}m_{\mathscr{A}}(A_i)\cdot\int_{A}\int_{\mathcal{X}}\chi(x\in E)\,f_a^X(x)\,\mathrm{d}x\,\mathrm{d}P_{A_i}(a),\quad P_{A_i}\in\mathcal{M}(A_i), i=1,\dots,n\right\}$$

with $\mathcal{M}(A_i)=\{P:\ P(A_i)=1\}$, $A_i\in\mathcal{A}$.

$\mathcal{M}_Z$ is obtained in a similar way.

Formulas for the upper probability of failure

Uncertainties modelled by ordinary sets A and B

Strong independence:

$$\overline{p}_f^{\mathrm{S}}=\sup_{\substack{a\in A\\b\in B}}p_f(a,b)\qquad(\star)$$

Epistemic irrelevance:

$$\overline{p}_f^{X\not\to Z}=\sup_{a\in A}\int_{\mathcal{X}}\overline{q}(x)\,f_a^X(x)\,\mathrm{d}x\qquad(\diamond)$$

$$\overline{q}(x)=\sup_{b\in B}\int_{\mathcal{Z}}\chi(h(x,z)\le 0)\,f_b^Z(z)\,\mathrm{d}z$$

Uncertainties modelled by random sets $\mathscr{A}$ and $\mathscr{B}$

Strong independence:

$$\overline{p}_f^{\mathrm{S}}=\sup_{\substack{a_r\in A_r, r=1,\dots,|\mathscr{A}|\\b_s\in B_s, s=1,\dots,|\mathscr{B}|}}\sum_{i=1}^{|\mathscr{A}|}\sum_{j=1}^{|\mathscr{B}|}m_{\mathscr{A}}(A_i)m_{\mathscr{B}}(B_j)p_f(a_i,b_j)$$

Epistemic irrelevance:

$$\overline{p}_f^{X\not\to Z}=\sum_{i=1}^{|\mathscr{A}|}m_{\mathscr{A}}(A_i)\sup_{a\in A_i}\int_{\mathcal{X}}\overline{q}(x)\,f_a^X(x)\,\mathrm{d}x$$

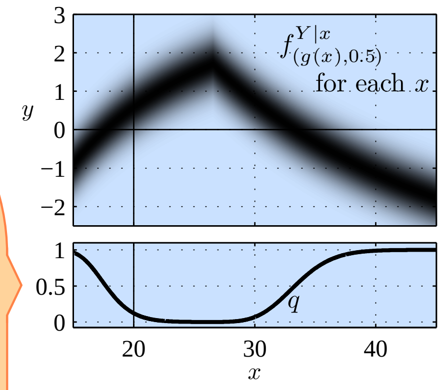
$$\overline{q}(x)=\sum_{j=1}^{|\mathscr{B}|}m_{\mathscr{B}}(B_j)\sup_{b\in B_j}\int_{\mathcal{Z}}\chi(h(x,z)\le 0)\,f_b^Z(z)\,\mathrm{d}z$$

Random set independence:

$$\overline{p}_f^{\mathrm{RS}}=\sum_{i=1}^{|\mathscr{A}|}\sum_{j=1}^{|\mathscr{B}|}m_{\mathscr{A}}(A_i)m_{\mathscr{B}}(B_j)\sup_{\substack{a\in A_i\\b\in B_j}}p_f(a,b),\ \text{cf. }(\star)$$

$$\overline{p}_f^{\mathrm{R},X\not\to Z}=\sum_{i=1}^{|\mathscr{A}|}\sum_{j=1}^{|\mathscr{B}|}m_{\mathscr{A}}(A_i)m_{\mathscr{B}}(B_j)\sup_{a\in A_i}\int_{\mathcal{X}}\overline{q}_j(x)\,f_a^X(x)\,\mathrm{d}x$$

$$\overline{q}_j(x)=\sup_{b\in B_j}\int_{\mathcal{Z}}\chi(h(x,z)\le 0)\,f_b^Z(z)\,\mathrm{d}z,\ \text{cf. }(\diamond)$$



Random limit state functions

Parameterization: $y=h(x,z)=g(x)+z$.

Uncertainty of z: f_b^Z density of a Gaussian distribution with $b=(\mu,\sigma)=(0,0.5)$.

Random variable $Y_{|x}$ for the output of the limit state function:

$Y_{|x}=h(x,Z)=g(x)+Z$ given x,

$Z\sim\mathcal{N}(\mu,\sigma^2)$, $Y_{|x}\sim\mathcal{N}(g(x)+\mu,\sigma^2)$.

Density $f^{Y|x}$ of $Y_{|x}$ is plotted for each x.

Independence of random variables X and Z

(a) *If we learn the values of the basic variables x, our knowledge about the parameters z and therefore about the choice of the limit state functions g_z does not change.*

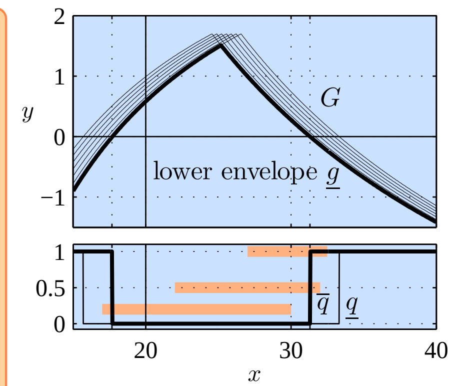
(b) *Learning the values of the parameters z and therefore learning which limit state function g_z to use has no influence on our knowledge about the variables x.*

Sets of parameterized limit state functions

Uncertainty of x: $f_a^X:=\delta_x$, $a:=x$, random set $\mathscr{A}$ given by three focal sets A_i and weights $m_{\mathscr{A}}(A_i)$.

Parameterization: $h(x,z)=g_z(x)=g(x+z)$.

Uncertainty of z: $f_b^Z:=\delta_z$, $b:=z$, $z\in B=[\underline{b},\overline{b}]$, $G=\{g_z: g_z(x)=h(x,z), z\in B\}$.

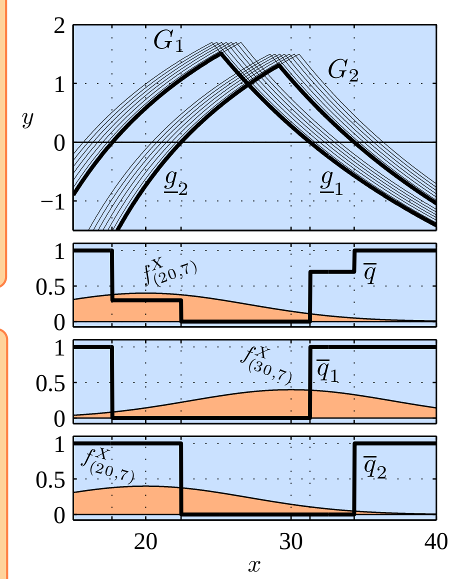


$$\overline{p}_f^{\mathrm{S}}=\sup_{\substack{x_r\in A_r, r=1,\dots,|\mathscr{A}|\\z\in B}}\sum_{i=1}^{|\mathscr{A}|}m_{\mathscr{A}}(A_i)p_f(x_i,z)$$

$$\overline{p}_f^{X\not\to Z}=\sum_{i=1}^{|\mathscr{A}|}m_{\mathscr{A}}(A_i)\sup_{x\in A_i}\chi(\underline{g}(x)\le 0)$$

$$\overline{p}_f^{\mathrm{RS}}=\overline{p}_f^{\mathrm{R},X\not\to Z}=\sum_{i=1}^{|\mathscr{A}|}m_{\mathscr{A}}(A_i)\sup_{\substack{x\in A_i\\z\in B}}p_f(x,z)$$

Random sets of parameterized limit state functions



Uncertainty of x: f_a^X is the density of a Gaussian distribution with parameters $a=(\mu,\sigma)$ where the parameter a is uncertain and modelled by the set

$$A=\{(\mu,\sigma):(\mu,\sigma)\in[20,30]\times\{7\}\}$$

Parameterization:

$$h(x,z)=g_z(x)=g(x+z_1)-z_2$$

Uncertainty of z:

$f_b^Z:=\delta_z$, $b:=z$, random set $\mathscr{B}$ given by two focal sets $B_1=[0,2]\times\{0\}$ and $B_2=[-4,-2]\times\{0.2\}$ with weights $m_{\mathscr{B}}(B_1)=0.7$ and $m_{\mathscr{B}}(B_2)=0.3$.

$G_1=\{g_z:\ g_z(x)=g(x+z_1)-z_2, z\in B_1\}$

$G_2=\{g_z:\ g_z(x)=g(x+z_1)-z_2, z\in B_2\}$

$$\overline{p}_f^{X\not\to Z}=\sup_{(\mu,\sigma)\in A}\int_{\mathcal{X}}\overline{q}(x)\,f_{(\mu,\sigma)}^X(x)\,\mathrm{d}x=\int_{\mathcal{X}}\overline{q}(x)\,f_{(20,7)}^X(x)\,\mathrm{d}x$$

$$=0.7\int_{\mathcal{X}}\overline{q}_1(x)\,f_{(20,7)}^X(x)\,\mathrm{d}x+0.3\int_{\mathcal{X}}\overline{q}_2(x)\,f_{(20,7)}^X(x)\,\mathrm{d}x=0.4929$$

$$\overline{p}_f^{\mathrm{RS}}=\sum_{j=1}^{|\mathscr{B}|}m_{\mathscr{B}}(B_j)\sup_{\substack{z\in B_j\\\mu\in[20,30]}}p_f((\mu,7),z)=0.7\sup_{z_1\in[0,2]}\int_{\mathcal{X}}\chi(g(x+z_1)\le 0)\,f_{(30,7)}^X(x)\,\mathrm{d}x$$

$$+0.3\sup_{z_1\in[-4,-2]}\int_{\mathcal{X}}\chi(g(x+z_1)-0.2\le 0)\,f_{(20,7)}^X(x)\,\mathrm{d}x=0.5063$$

$$\overline{p}_f^{\mathrm{R},X\not\to Z}=\sum_{j=1}^{|\mathscr{B}|}m_{\mathscr{B}}(B_j)\sup_{\mu\in[20,30]}\int_{\mathcal{X}}\overline{q}_j(x)\,f_{(\mu,7)}^X(x)\,\mathrm{d}x$$

$$=0.7\int_{\mathcal{X}}\overline{q}_1(x)\,f_{(30,7)}^X(x)\,\mathrm{d}x+0.3\int_{\mathcal{X}}\overline{q}_2(x)\,f_{(20,7)}^X(x)\,\mathrm{d}x=0.5225$$

Summary

- **Ordering:** $\overline{p}_f^{\mathrm{S}}\le\overline{p}_f^{X\not\to Z}\le\overline{p}_f^{\mathrm{R},X\not\to Z}$ and $\overline{p}_f^{\mathrm{S}}\le\overline{p}_f^{\mathrm{RS}}\le\overline{p}_f^{\mathrm{R},X\not\to Z}$.
- **Strong independence:** Complete information about the uncertain limit state function is needed.
- **Epistemic irrelevance:** Sufficient to know $\overline{q}$ which condenses the uncertain limit state function. $\overline{q}$ could be a starting point for uncertainty modelling.
- **Random set independence:** Possibility to combine upper probabilities of failure resulting from different and independent computations.
- The amount of information to deal with decreases from the uncertain limit state function itself to the function $\overline{q}$ and to the upper probabilities.