Independent natural extension for sets of desirable gambles

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Abstract

We investigate how to combine a number of marginal coherent sets of desirable gambles into a joint set using the properties of epistemic irrelevance and independence. We provide formulas for the smallest such joint, called their independent natural extension, and study its main properties. The independent natural extension of maximal sets of gambles allows us to define the strong product of sets of desirable gambles. Finally, we explore an easy way to generalise these results to also apply for the conditional versions of epistemic irrelevance and independence.

Keywords. Epistemic irrelevance, epistemic independence, independent natural extension, strong product, coherent set of desirable gambles.

1 Introduction

One disadvantage of working with coherent lower previsions (or previsions and probabilities for that matter), is that conditioning a lower prevision does not necessarily lead to uniquely coherent results when the conditioning event has lower probability zero; see for instance Ref. [8, Section 6.4]. For precise probabilities, this difficulty can be circumvented by using full conditional measures [5]. In an imprecise-probabilities context, working with the more informative coherent sets of desirable gambles rather than with lower previsions provides a very elegant and intuitively appealing way out of this problem, as Walley already suggested in 1991 [8, Section 3.8.6 and Appendix F], and argued in much more detail in 2000 [9]. The connection between full conditional measures and maximal coherent sets of desirable gambles was explored by Couso and Moral [1]. De Cooman and Quaeghebeur [4] have shown that working with sets of desirable gambles is especially illuminating in the context of modelling exchangeability assessments.

Exchangeability is a structural assessment, and so is independence. Conditioning and independence are, of course, closely related. In a recent paper [3], we investigated the notions of epistemic independence of finite-valued variables using coherent lower previsions. The above-mentioned problems with conditioning, and the fact that the coherence requirements for conditional lower previsions are, to be honest, quite cumbersome to work with, have turned this into a quite complicated exercise. This is the reason why, in the present paper, we investigate if looking at independence using sets of desirable gambles leads to a more elegant theory that avoids some of the complexity pitfalls of working with coherent lower previsions. In doing this, we build on the strong pioneering work on epistemic irrelevance by Moral [7]. While we focus here on the symmetrised notion of epistemic independence, much of what we do can be seen as an application and continuation of his ideas.

In Section 2 we summarise relevant results in the existing theory of sets of desirable gambles. After mentioning useful notational conventions in Section 3, we recall the basic marginalisation, conditioning and extension operations for sets of desirable gambles in Sections 4 and 5. We use these to combine a number of marginal sets of desirable gambles into a joint satisfying epistemic irrelevance (Section 6), and epistemic independence (Section 7). In Section 8, we study the particular case of maximal sets of desirable gambles, and derive the concept of a strong product. Section 9 deals with conditional independence assessments.

2 Coherent sets of desirable gambles and natural extension

Consider a variable X taking values in some non-empty set $\mathscr{X}$, that we shall assume to be finite. We model information about X by means of sets of desirable gambles. A *gamble* is a real-valued function on $\mathscr{X}$, and we denote the set of all gambles on $\mathscr{X}$ by $\mathscr{G}(\mathscr{X})$. It is a linear space under point-wise addition of gambles and point-wise multiplication of gambles with real numbers. For any subset $\mathscr{A}$ of $\mathscr{G}(\mathscr{X})$, we denote by $\operatorname{posi}(\mathscr{A})$ the set of all positive linear combinations of gambles in $\mathscr{A}$:

$$\operatorname{posi}(\mathscr{A}) := \left\{ \sum_{k=1}^{n} \lambda_k f_k \colon f_k \in \mathscr{A}, \lambda_k > 0, n > 0 \right\}.$$

We call $\mathscr{A}$ a *convex cone* if it is closed under positive linear combinations, meaning that $\operatorname{posi}(\mathscr{A}) = \mathscr{A}$.

For any gambles f and g on $\mathscr{X}$, we write '$f \geq g$' if $(\forall x \in \mathscr{X}) f(x) \geq g(x)$, and '$f > g$' if $f \geq g$ and $f \neq g$. A gamble $f > 0$ is called *positive*. A gamble $g \leq 0$ is called *non-positive*. $\mathscr{G}(\mathscr{X})_{\neq 0}$ denotes the set of all non-zero gambles, $\mathscr{G}(\mathscr{X})_{>0}$ the convex cone of all positive gambles, and $\mathscr{G}(\mathscr{X})_{\leq 0}$ the convex cone of all non-positive gambles.

2.1 Coherence and avoiding non-positivity

Definition 1 ([4]). *A set of desirable gambles $\mathscr{D} \subseteq \mathscr{G}(\mathscr{X})$* avoids non-positivity *if $\mathscr{G}(\mathscr{X})_{\leq 0} \cap \operatorname{posi}(\mathscr{D}) = \emptyset$. It is called* coherent *if:*

D1. $0 \notin \mathscr{D}$;
D2. $\mathscr{G}(\mathscr{X})_{>0} \subseteq \mathscr{D}$;
D3. $\mathscr{D} = \operatorname{posi}(\mathscr{D})$.

We denote by $\mathbb{D}(\mathscr{X})$ the set of all coherent sets of desirable gambles on $\mathscr{X}$.

Requirement D3 turns $\mathscr{D}$ into a convex cone. Due to D2, it includes $\mathscr{G}(\mathscr{X})_{>0}$; due to D1–D3, it excludes $\mathscr{G}(\mathscr{X})_{\leq 0}$, and therefore avoids non-positivity.

2.2 Natural extension

If we consider any non-empty family of coherent sets of desirable gambles $\mathscr{D}_i$, $i \in I$, then their intersection $\bigcap_{i \in I} \mathscr{D}_i$ is still coherent. This is the idea behind the following result. If a subject gives us an *assessment*, a set $\mathscr{A} \subseteq \mathscr{G}(\mathscr{X})$ of gambles on $\mathscr{X}$ that he finds desirable, then we can tell exactly when this assessment can be extended to a coherent set, and how to construct the smallest such set.

Theorem 1 (Natural extension [4]). *Consider an assessment $\mathscr{A} \subseteq \mathscr{G}(\mathscr{X})$, and define its* natural extension *as:*[1]

$$\mathscr{E}(\mathscr{A}) := \bigcap \{\mathscr{D} \in \mathbb{D}(\mathscr{X}) \colon \mathscr{A} \subseteq \mathscr{D}\}$$

Then the following statements are equivalent:

(i) *$\mathscr{A}$ avoids non-positivity;*
(ii) *$\mathscr{A}$ is included in some coherent set of desirable gambles;*
(iii) *$\mathscr{E}(\mathscr{A}) \neq \mathscr{G}(\mathscr{X})$;*
(iv) *the set of desirable gambles $\mathscr{E}(\mathscr{A})$ is coherent;*
(v) *$\mathscr{E}(\mathscr{A})$ is the smallest coherent set of desirable gambles that includes $\mathscr{A}$.*

When any (and hence all) of these equivalent statements hold, then $\mathscr{E}(\mathscr{A}) = \operatorname{posi}\big(\mathscr{G}(\mathscr{X})_{>0} \cup \mathscr{A}\big)$.

2.3 Helpful lemmas

In order to prove a number of results in this paper, we need the following lemmas, one of which is convenient version of the separating hyperplane theorem:

Lemma 2. *Consider a finite subset $\mathscr{A}$ of $\mathscr{G}(\mathscr{X})$. Then $0 \notin \operatorname{posi}(\mathscr{G}(\mathscr{X})_{>0} \cup \mathscr{A})$ if and only if there is some probability mass function p such that $\sum_{x \in \mathscr{X}} p(x) f(x) > 0$ for all $f \in \mathscr{A}$ and $p(x) > 0$ for all $x \in \mathscr{X}$.*

Proof. It clearly suffices to prove necessity. Since $0 \notin \operatorname{posi}(\mathscr{G}(\mathscr{X})_{>0} \cup \mathscr{A})$, we infer from a version of the separating hyperplane theorem [8, Appendix E.1] that there is a linear functional Λ on $\mathscr{G}(\mathscr{X})$ such that

$$(\forall x \in \mathscr{X}) \Lambda(\mathbb{I}_{\{x\}}) > 0 \text{ and } (\forall f \in \mathscr{A}) \Lambda(f) > 0.$$

Then $\Lambda(\mathscr{X}) = \sum_{x \in \mathscr{X}} \Lambda(\mathbb{I}_{\{x\}}) > 0$, and if we let $p(x) := \Lambda(\mathbb{I}_{\{x\}}) / \Lambda(\mathscr{X}) > 0$ for all $x \in \mathscr{X}$, then p is a probability mass function on $\mathscr{X}$ for which $\Lambda(f)/\Lambda(\mathscr{X}) = \sum_{x \in \mathscr{X}} p(x) f(x) > 0$ for all $f \in \mathscr{A}$. □

Lemma 3. *Consider a convex cone $\mathscr{A}$ of gambles on $\mathscr{X}$ such that $\max f > 0$ for all $f \in \mathscr{A}$. Consider any non-zero gamble g on $\mathscr{X}$. If $g \notin \mathscr{A}$ then $0 \notin \operatorname{posi}(\mathscr{A} \cup \{-g\})$.*

Proof. Consider a non-zero gamble $g \notin \mathscr{A}$, and assume *ex absurdo* that $0 \in \operatorname{posi}(\mathscr{A} \cup \{-g\})$. Then it follows from the assumptions that there are $f \in \mathscr{A}$ and $\mu > 0$ such that $0 = f + \mu(-g)$. Hence $g \in \mathscr{A}$, a contradiction. □

2.4 Maximal sets of desirable gambles

An element $\mathscr{D}$ of $\mathbb{D}(\mathscr{X})$ is called *maximal* if it is not strictly included in any other element of $\mathbb{D}(\mathscr{X})$, or in other words, if adding any gamble f to $\mathscr{D}$ makes sure we can no longer extend the set $\mathscr{D} \cup \{f\}$ to a set that is still coherent:

$$(\forall \mathscr{D}' \in \mathbb{D}(\mathscr{X}))(\mathscr{D} \subseteq \mathscr{D}' \Rightarrow \mathscr{D} = \mathscr{D}')$$

$\mathbb{M}(\mathscr{X})$ denotes the set of all maximal elements of $\mathbb{D}(\mathscr{X})$.

The following proposition provides a characterisation of such maximal elements.

Proposition 4 ([1, 4]). *Let $\mathscr{D} \in \mathbb{D}(\mathscr{X})$, then $\mathscr{D}$ is a maximal coherent set of desirable gambles if and only if*

$$(\forall f \in \mathscr{G}(\mathscr{X})_{\neq 0})(f \notin \mathscr{D} \Rightarrow -f \in \mathscr{D}).$$

For the following important result, it is easy to provide a constructive proof, based on the same ideas as in Ref. [1]. For the more general case of infinite $\mathscr{X}$, a non-constructive proof can be based on Zorn's Lemma [4].

Theorem 5 ([1, 4]). *A subset $\mathscr{A}$ of $\mathscr{G}(\mathscr{X})$ avoids non-positivity if and only if $m(\mathscr{A}) := \{\mathscr{M} \in \mathbb{M}(\mathscr{X}) \colon \mathscr{A} \subseteq \mathscr{M}\}$ is non-empty. Moreover, $\mathscr{E}(\mathscr{A}) = \bigcap m(\mathscr{A})$.*

2.5 Coherent lower previsions

Given a coherent set of desirable gambles $\mathscr{D}$, the functional $\underline{P}$ defined on $\mathscr{G}(\mathscr{X})$ by

$$\underline{P}(f) := \sup \{\mu \colon f - \mu \in \mathscr{D}\} \tag{1}$$

[1] As usual, in this expression, we let $\bigcap \emptyset = \mathscr{G}(\mathscr{X})$.

is a coherent lower prevision [8, Theorem 3.8.1], and therefore corresponds to taking a lower envelope of expectations with respect a set of probability mass functions. Many different coherent sets of desirable gambles induce the same coherent lower prevision $\underline{P}$. The smallest is called the associated *set of strictly desirable* gambles:

$$\mathscr{D}' := \{f \in \mathscr{G}(\mathscr{X}) \colon f > 0 \text{ or } \underline{P}(f) > 0\}. \qquad (2)$$

When $\mathscr{D}$ is a maximal coherent set of desirable gambles, the lower prevision $\underline{P}$ defined by Eq. (1) is a *linear prevision*, meaning that it corresponds to an expectation operator with respect to a probability mass function. For more information, see Refs. [1, Section 5], [6, Proposition 6], [8] and [10].

3 Basic notation

From now on we consider a number of variables X_n, $n \in N$, taking values in the respective finite sets $\mathscr{X}_n$. Here N is some finite non-empty index set.

For every subset R of N, we denote by X_R the tuple of variables (with one component for each $r \in R$) that takes values in the Cartesian product $\mathscr{X}_R := \times_{r\in R}\mathscr{X}_r$. The elements of $\mathscr{X}_R$ are generically denoted by x_R or z_R, with corresponding components $x_r := x_R(r)$ or $z_r := z_R(r)$, $r \in R$.

We will assume that the variables X_n are logically independent, which means that for each subset R of N, X_R may assume all values in $\mathscr{X}_R$.

We denote by $\mathscr{G}(\mathscr{X}_R)$ the set of gambles defined on $\mathscr{X}_R$. We will frequently resort to the simplifying device of *identifying* a gamble on $\mathscr{X}_R$ with a gamble on $\mathscr{X}_N$, namely its cylindrical extension. To give an example, if $\mathscr{K} \subseteq \mathscr{G}(\mathscr{X}_N)$, this trick allows us to consider $\mathscr{K} \cap \mathscr{G}(\mathscr{X}_R)$ as the set of those gambles in $\mathscr{K}$ that depend only on the variable X_R. As another example, this device allows us to identify the gambles $\mathbb{I}_{\{x_R\}}$ and $\mathbb{I}_{\{x_R\}\times\mathscr{X}_{N\setminus R}}$, and therefore also the events $\{x_R\}$ and $\{x_R\} \times \mathscr{X}_{N\setminus R}$. More generally, for any event $A \subseteq \mathscr{X}_R$, we can identify the gambles $\mathbb{I}_A$ and $\mathbb{I}_{A\times\mathscr{X}_{N\setminus R}}$, and therefore also the events A and $A \times \mathscr{X}_{N\setminus R}$.

We draw attention to the case $R = \emptyset$. By definition, $\mathscr{X}_\emptyset$ contains only one element $x_\emptyset$: the empty map $\emptyset \to \emptyset$. There is no uncertainty about the value of the variable $X_\emptyset$: it can assume only one value (the empty map), and $\mathbb{I}_{\mathscr{X}_\emptyset} = \mathbb{I}_{\{x_\emptyset\}} = 1$. We can identify $\mathscr{G}(\mathscr{X}_\emptyset)$ with the set of real numbers $\mathbb{R}$. There is only one coherent set of desirable gambles on $\mathscr{X}_\emptyset$: the set $\mathbb{R}_{>0}$ of positive real numbers.

4 Marginalisation and cylindrical extension

Suppose that we have a set $\mathscr{D}_N \subseteq \mathscr{G}(\mathscr{X}_N)$ of desirable gambles modelling a subject's information about the uncertain variable X_N. We are interested in modelling the information about the variable X_O, where O is some subset of N. This can be done using the set of desirable gambles that belong to $\mathscr{D}_N$ but only depend on the variable X_O:

$$\mathrm{marg}_O(\mathscr{D}_N) := \{g \in \mathscr{G}(\mathscr{X}_O) \colon g \in \mathscr{D}_N\} = \mathscr{D}_N \cap \mathscr{G}(\mathscr{X}_O) \qquad (3)$$

is called a *marginal* set of desirable gambles [7]. Observe that $\mathrm{marg}_\emptyset(\mathscr{D}_N) = \mathscr{G}(\mathscr{X}_\emptyset)_{>0}$, which can be identified with the set of positive real numbers $\mathbb{R}_{>0}$. Also, with $O_1, O_2 \subseteq N$, it is obvious that

$$O_1 \subseteq O_2 \Rightarrow \mathrm{marg}_{O_1}(\mathrm{marg}_{O_2}(\mathscr{D}_N)) = \mathrm{marg}_{O_1}(\mathscr{D}_N). \qquad (4)$$

Coherence is trivially preserved under marginalisation:

Proposition 6. *Let $\mathscr{D}_N$ be a set of desirable gambles on $\mathscr{X}_N$, and consider any subset O of N.*

(i) *If $\mathscr{D}_N$ avoids non-positivity, then so does* $\mathrm{marg}_O(\mathscr{D}_N)$.
(ii) *If $\mathscr{D}_N$ is coherent, then* $\mathrm{marg}_O(\mathscr{D}_N)$ *is a coherent set of desirable gambles on $\mathscr{X}_O$.*

We now look for a kind of inverse operation to marginalisation. Suppose we have a coherent set $\mathscr{D}_O \subseteq \mathscr{G}(\mathscr{X}_O)$ of desirable gambles modelling a subject's information about the uncertain variable X_O, and we want to extend this to a coherent set of desirable gambles on $\mathscr{X}_N$, representing the same information. So we are looking for a coherent set of desirable gambles $\mathscr{D}_N \subseteq \mathscr{G}(\mathscr{X}_N)$ such that $\mathrm{marg}_O(\mathscr{D}_N) = \mathscr{D}_O$ and that is as small as possible: the most conservative coherent set of desirable gambles on $\mathscr{X}_N$ that marginalises to $\mathscr{D}_O$.

Proposition 7. *Let O be a subset of N and let $\mathscr{D}_O \in \mathbb{D}(\mathscr{X}_O)$. Then the most conservative (smallest) coherent set of desirable gambles on $\mathscr{X}_N$ that marginalises to $\mathscr{D}_O$ is given by*

$$\mathrm{ext}_N(\mathscr{D}_O) := \mathrm{posi}(\mathscr{G}(\mathscr{X}_N)_{>0} \cup \mathscr{D}_O). \qquad (5)$$

It is called the cylindrical extension *of $\mathscr{D}_O$ to a set of desirable gambles on $\mathscr{X}_N$, and satisfies*

$$\mathrm{marg}_O(\mathrm{ext}_N(\mathscr{D}_O)) = \mathscr{D}_O. \qquad (6)$$

This extension is called *weak extension* by Moral [7, Section 2.1].

Proof. It is clear from the coherence requirements and Eq. (3) that any coherent set that marginalises to $\mathscr{D}_O$ must include $\mathscr{G}(\mathscr{X}_N)_{>0}$ and $\mathscr{D}_O$, and therefore also $\mathrm{posi}(\mathscr{G}(\mathscr{X}_N)_{>0} \cup \mathscr{D}_O) = \mathrm{ext}_N(\mathscr{D}_O)$. It therefore suffices to prove that $\mathrm{posi}(\mathscr{G}(\mathscr{X}_N)_{>0} \cup \mathscr{D}_O)$ is coherent, and that it marginalises to $\mathscr{D}_O$.

To prove coherence, it suffices to prove that $\mathscr{D}_O$ avoids non-positivity, by Theorem 1. But this is obvious because $\mathscr{D}_O$ is a coherent set of desirable gambles on $\mathscr{X}_O$.

We are left to prove that $\mathrm{marg}_O(\mathrm{ext}_N(\mathscr{D}_O)) = \mathscr{D}_O$. Since for any $g \in \mathscr{D}_O$ it is obvious that both $g \in \mathrm{ext}_N(\mathscr{D}_O)$ and $g \in \mathscr{G}(\mathscr{X}_O)$, we see immediately that $\mathscr{D}_O \subseteq \mathrm{marg}_O(\mathrm{ext}_N(\mathscr{D}_O))$, so we concentrate on proving that $\operatorname{marg}_O(\operatorname{ext}_N(\mathscr{D}_O)) \subseteq \mathscr{D}_O$. Consider $f \in \operatorname{marg}_O(\operatorname{ext}_N(\mathscr{D}_O))$, meaning that both $f \in \mathscr{G}(\mathscr{X}_O)$ and $f \in \operatorname{ext}_N(\mathscr{D}_O)$. The latter means that there are $g \in \mathscr{D}_O$, $h \in \mathscr{G}(\mathscr{X}_N)_{>0}$, and non-negative λ and μ such that $\max\{\lambda, \mu\} > 0$ for which $f = \lambda g + \mu h$. Since we need to prove that $f \in \mathscr{D}_O$, we can assume without loss of generality that $\mu > 0$. But then $h = (f - \lambda g)/\mu \in \mathscr{G}(\mathscr{X}_O)$ and therefore also $h \in \mathscr{G}(\mathscr{X}_O)_{>0}$, whence indeed $f \in \mathscr{D}_O$, by coherence of $\mathscr{D}_O$. □

5 Conditioning

Suppose that we have a set $\mathscr{D}_N \subseteq \mathscr{G}(\mathscr{X}_N)$ of desirable gambles modelling a subject's information about the uncertain variable X_N. Consider a subset I of N, and assume we want to update the model $\mathscr{D}_N$ with the information that $X_I = x_I$. This leads to an updated set of desirable gambles:

$$\mathscr{D}_N | x_I := \left\{ f \in \mathscr{G}(\mathscr{X}_N) \colon \mathbb{I}_{\{x_I\}} f \in \mathscr{D}_N \right\}. \tag{7}$$

For technical reasons, and mainly in order to streamline the proofs as much as possible, we also allow the admittedly pathological case that $I = \emptyset$. Since $\mathbb{I}_{\{x_\emptyset\}} = 1$, this amounts to not conditioning at all.

Eq. (7) introduces the conditioning operator '$|$' essentially used by Walley [9] and Moral [7]. We prefer a slightly modified version '$\rfloor$' [4]. Since $\mathbb{I}_{\{x_I\}} f = \mathbb{I}_{\{x_I\}} f(x_I, \cdot)$, we can characterise the updated model $\mathscr{D}_N | x_I$ through the set

$$\mathscr{D}_N \rfloor x_I := \left\{ g \in \mathscr{G}(\mathscr{X}_{N \setminus I}) \colon \mathbb{I}_{\{x_I\}} g \in \mathscr{D}_N \right\} \subseteq \mathscr{G}(\mathscr{X}_{N \setminus I}),$$

in the specific sense that for all $g \in \mathscr{G}(\mathscr{X}_{N \setminus I})$:

$$g \in \mathscr{D}_N \rfloor x_I \Leftrightarrow \mathbb{I}_{\{x_I\}} g \in \mathscr{D}_N \Leftrightarrow \mathbb{I}_{\{x_I\}} g \in \mathscr{D}_N | x_I, \tag{8}$$

and for all $f \in \mathscr{G}(\mathscr{X}_N)$: $f \in \mathscr{D}_N | x_I \Leftrightarrow f(x_I, \cdot) \in \mathscr{D}_N \rfloor x_I$. Coherence is trivially preserved under conditioning:

Proposition 8. *Let $\mathscr{D}_N$ be a coherent set of desirable gambles on $\mathscr{X}_N$, and consider any subset I of N. Then $\mathscr{D}_N \rfloor x_I$ is a coherent set of desirable gambles on $\mathscr{X}_{N \setminus I}$.*

The order of marginalisation and conditioning can be reversed, under some conditions.

Proposition 9. *Let $\mathscr{D}_N$ be a coherent set of desirable gambles on $\mathscr{X}_N$, and consider any disjoint subsets I and O of N. Then $\operatorname{marg}_O(\mathscr{D}_N \rfloor x_I) = \operatorname{marg}_{I \cup O}(\mathscr{D}_N) \rfloor x_I$ for all $x_I \in \mathscr{X}_I$.*

Proof. Consider any $h \in \mathscr{G}(\mathscr{X}_N)$ and observe the following chain of equivalences:

$$\begin{aligned} h \in \operatorname{marg}_O(\mathscr{D}_N \rfloor x_I) &\Leftrightarrow h \in \mathscr{G}(\mathscr{X}_O) \text{ and } h \in \mathscr{D}_N \rfloor x_I \\ &\Leftrightarrow h \in \mathscr{G}(\mathscr{X}_O) \text{ and } \mathbb{I}_{\{x_I\}} h \in \mathscr{D}_N \\ &\Leftrightarrow h \in \mathscr{G}(\mathscr{X}_O) \text{ and } \mathbb{I}_{\{x_I\}} h \in \operatorname{marg}_{I \cup O}(\mathscr{D}_N) \\ &\Leftrightarrow h \in \mathscr{G}(\mathscr{X}_O) \text{ and } h \in \operatorname{marg}_{I \cup O}(\mathscr{D}_N) \rfloor x_I \\ &\Leftrightarrow h \in \operatorname{marg}_{I \cup O}(\mathscr{D}_N) \rfloor x_I. \end{aligned}$$

□

6 Irrelevant natural extension

We are now ready to look at the simplest type of irrelevance judgement. Consider two disjoint subsets I and O of N. We say that X_I is *epistemically irrelevant* to X_O when learning the value of X_I does not influence or change our subject's beliefs about X_O.

When does a set $\mathscr{D}_N$ of desirable gambles on $\mathscr{X}_N$ capture this type of epistemic irrelevance? Observing that $X_I = x_I$ turns $\mathscr{D}_N$ into the updated set $\mathscr{D}_N \rfloor x_I$ of desirable gambles on $\mathscr{X}_{N \setminus I}$, we should clearly require that:

$$\operatorname{marg}_O(\mathscr{D}_N \rfloor x_I) = \operatorname{marg}_O(\mathscr{D}_N) \text{ for all } x_I \in \mathscr{X}_I. \tag{9}$$

As before, for technical reasons we also allow I and O to be empty. It is clear from the definition above that the 'variable' $X_\emptyset$, about whose constant value we are certain, is epistemically irrelevant to any variable X_O. Similarly, we see that any variable X_I is epistemically irrelevant to the 'variable' $X_\emptyset$. This seems to be in accordance with intuition.

The epistemic irrelevance condition can be formulated trivially in an interesting and slightly different manner.

Proposition 10. *Let $\mathscr{D}_N$ be a coherent set of desirable gambles on $\mathscr{X}_N$, and let I and O be any disjoint subsets of N. Then the following statements are equivalent:*

(i) $\operatorname{marg}_O(\mathscr{D}_N \rfloor x_I) = \operatorname{marg}_O(\mathscr{D}_N)$ *for all $x_I \in \mathscr{X}_I$;*
(ii) *for all $f \in \mathscr{G}(\mathscr{X}_O)$ and all $x_I \in \mathscr{X}_I$: $\mathbb{I}_{\{x_I\}} f \in \mathscr{D}_N \Leftrightarrow f \in \mathscr{D}_N$.*

Irrelevance assessments are most useful in constructing sets of desirable gambles from other ones. Suppose we have a coherent set $\mathscr{D}_O$ of desirable gambles on $\mathscr{X}_O$, and an assessment that X_I is epistemically irrelevant to X_O, where I and O are disjoint index sets. Then how can we combine $\mathscr{D}_O$ and this structural irrelevance assessment into a coherent set of desirable gambles on $\mathscr{X}_{I \cup O}$, or more generally, on $\mathscr{X}_N$, where $N \supseteq I \cup O$? To see how this can be done in a way that is as conservative as possible, we introduce:

$$\mathscr{A}^{\text{irr}}_{I \to O} := \operatorname{posi}\left(\left\{ \mathbb{I}_{\{x_I\}} g \colon g \in \mathscr{D}_O \text{ and } x_I \in \mathscr{X}_I \right\}\right).$$

It follows from the next lemma that for all $h \in \mathscr{G}(\mathscr{X}_{I \cup O})$:

$$h \in \mathscr{A}^{\text{irr}}_{I \to O} \Leftrightarrow h \neq 0 \text{ and } (\forall x_I \in \mathscr{X}_I) h(x_I, \cdot) \in \mathscr{D}_O \cup \{0\}. \tag{10}$$

Clearly, and this will be quite important in streamlining proofs, $\mathscr{A}^{\text{irr}}_{\emptyset \to O} = \mathscr{D}_O$ and $\mathscr{A}^{\text{irr}}_{I \to \emptyset} = \mathscr{G}(\mathscr{X}_I)_{>0}$. We also give two important properties of these sets:

Lemma 11. *Consider disjoint subsets I and O of N, and a coherent set $\mathscr{D}_O$ of desirable gambles on $\mathscr{X}_O$. Then $\mathscr{A}^{\text{irr}}_{I \to O}$ is a coherent set of desirable gambles on $\mathscr{X}_{I \cup O}$.*

Proof. D1. Assume *ex absurdo* that there are $n > 0$, real $\lambda_k > 0$ and $f_k \in \mathscr{A}^{\text{irr}}_{I \to O}$ such that $\sum_{k=1}^n \lambda_k f_k = 0$. It follows from the assumptions that there are $\ell \in \{1, \ldots, n\}$ and $x_I \in \mathscr{X}_I$ such that $f_\ell(x_I, \cdot) \neq 0$. This implies that in the sum $\sum_{k=1}^n \lambda_k f_k(x_I, \cdot) = 0$

not all the gambles $\lambda_k f_k(x_I,\cdot)$ are zero. Since the non-zero ones belong to $\mathscr{D}_O$, this contradicts the coherence of $\mathscr{D}_O$.

D2. Consider any $h \in \mathscr{G}(\mathscr{X}_{I\cup O})_{>0}$. Then clearly $h(x_I,\cdot) \geq 0$ and therefore $h(x_I,\cdot) \in \mathscr{D}_O \cup \{0\}$ for all $x_I \in \mathscr{X}_I$. Since $h \neq 0$, it follows that indeed $h \in \mathscr{A}^{\text{irr}}_{I\to O}$.

D3. Trivial if we recall that $\text{posi}(\text{posi}(\mathscr{D})) = \text{posi}(\mathscr{D})$ for any set of desirable gambles $\mathscr{D}$. □

Lemma 12. *Consider disjoint subsets I and O of N, and a coherent set $\mathscr{D}_O$ of desirable gambles on $\mathscr{X}_O$. Then* $\text{marg}_O(\mathscr{A}^{\text{irr}}_{I\to O}) = \mathscr{D}_O$.

Proof. It is obvious from Eq. (10) that indeed:

$$\begin{aligned}\text{marg}_O(\mathscr{A}^{\text{irr}}_{I\to O}) &= \mathscr{A}^{\text{irr}}_{I\to O} \cap \mathscr{G}(\mathscr{X}_O)\\ &= \left\{h \in \mathscr{G}(\mathscr{X}_O)_{\neq 0} \colon (\forall x_I \in \mathscr{X}_I) h \in \mathscr{D}_O \cup \{0\}\right\}\\ &= \left\{h \in \mathscr{G}(\mathscr{X}_O)_{\neq 0} \colon h \in \mathscr{D}_O \cup \{0\}\right\} = \mathscr{D}_O. \quad \square\end{aligned}$$

Theorem 13. *Consider disjoint subsets I and O of N, and a coherent set $\mathscr{D}_O$ of desirable gambles on $\mathscr{X}_O$. Then the smallest coherent set of desirable gambles on $\mathscr{X}_N$ that marginalises to $\mathscr{D}_O$ and satisfies the epistemic irrelevance condition* (9) *of X_I to X_O is given by* $\text{ext}_N(\mathscr{A}^{\text{irr}}_{I\to O}) = \text{posi}(\mathscr{G}(\mathscr{X}_N)_{>0} \cup \mathscr{A}^{\text{irr}}_{I\to O})$.

Proof. Consider any coherent set $\mathscr{D}_N$ on $\mathscr{X}_N$ that marginalises to $\mathscr{D}_O$ and satisfies the irrelevance condition (9). This implies that $\text{marg}_O(\mathscr{D}_N\rfloor x_I) = \mathscr{D}_O$ for any $x_I \in \mathscr{X}_I$, so $g \in \mathscr{D}_N\rfloor x_I$, and therefore $\mathbb{I}_{\{x_I\}} g \in \mathscr{D}_N$ for any $g \in \mathscr{D}_O$, by Eq. (8). So we infer by coherence that $\mathscr{A}^{\text{irr}}_{I\to O} \subseteq \mathscr{D}_N$, and therefore also that $\text{posi}(\mathscr{G}(\mathscr{X}_N)_{>0} \cup \mathscr{A}^{\text{irr}}_{I\to O}) \subseteq \mathscr{D}_N$. As a consequence, it suffices to prove that (i) $\text{ext}_N(\mathscr{A}^{\text{irr}}_{I\to O})$ is coherent, (ii) marginalises to $\mathscr{D}_O$, and (iii) satisfies the epistemic irrelevance condition (9). This is what we now set out to do.

(i). By Lemma 11, $\mathscr{A}^{\text{irr}}_{I\to O}$ is a coherent set of desirable gambles on $\mathscr{X}_{I\cup O}$, so Proposition 7 implies that $\text{posi}(\mathscr{G}(\mathscr{X}_N)_{>0} \cup \mathscr{A}^{\text{irr}}_{I\to O}) = \text{ext}_N(\mathscr{A}^{\text{irr}}_{I\to O})$ is a coherent set of desirable gambles on $\mathscr{X}_N$.

(ii). Marginalisation leads to:

$$\begin{aligned}\text{marg}_O(\text{ext}_N(\mathscr{A}^{\text{irr}}_{I\to O})) &= \text{marg}_O(\text{marg}_{I\cup O}(\text{ext}_N(\mathscr{A}^{\text{irr}}_{I\to O})))\\ &= \text{marg}_O(\mathscr{A}^{\text{irr}}_{I\to O}) = \mathscr{D}_O,\end{aligned}$$

where the first equality follows from Eq. (4), the second from Eq. (6), and the third from Lemma 12.

(iii). It follows from Proposition 9 and Eq. (6) that

$$\begin{aligned}\text{marg}_O(\text{ext}_N(\mathscr{A}^{\text{irr}}_{I\to O})\rfloor x_I) &= \text{marg}_{I\cup O}(\text{ext}_N(\mathscr{A}^{\text{irr}}_{I\to O}))\rfloor x_I\\ &= \mathscr{A}^{\text{irr}}_{I\to O}\rfloor x_I,\end{aligned}$$

and we have just shown in (ii) that $\text{marg}_O(\text{ext}_N(\mathscr{A}^{\text{irr}}_{I\to O})) = \mathscr{D}_O$, so proving that $\text{marg}_O(\text{ext}_N(\mathscr{A}^{\text{irr}}_{I\to O})\rfloor x_I) = \text{marg}_O(\text{ext}_N(\mathscr{A}^{\text{irr}}_{I\to O}))$ amounts to proving that $\mathscr{A}^{\text{irr}}_{I\to O}\rfloor x_I = \mathscr{D}_O$. It is obvious from the definition of $\mathscr{A}^{\text{irr}}_{I\to O}$ that $\mathscr{D}_O \subseteq \mathscr{A}^{\text{irr}}_{I\to O}\rfloor x_I$, so we concentrate on the converse inclusion. Consider any $h \in \mathscr{A}^{\text{irr}}_{I\to O}\rfloor x_I$; then $\mathbb{I}_{\{x_I\}} h \in \mathscr{A}^{\text{irr}}_{I\to O}$, so we infer from Eq. (10) that in particular $h \in \mathscr{D}_O \cup \{0\}$. But since $\mathscr{A}^{\text{irr}}_{I\to O}$ is coherent by Lemma 11, we see that $h \neq 0$ and therefore indeed $h \in \mathscr{D}_O$. □

Theorem 13 is mentioned briefly, with only a hint at the proof, by Moral [7, Section 2.4]. We believe the result is not so trivial and have therefore decided to include our version of the proof here. Our notion of epistemic irrelevance is called *weak* epistemic irrelevance by Moral. For his version of epistemic irrelevance he requires in addition that $\mathscr{D}_N$ should be equal to the irrelevant natural extension of $\mathscr{D}_O$, and therefore be the *smallest* model that satisfies the (weak) epistemic irrelevance condition (9). While we feel comfortable with his reasons for doing so, we have decided not to follow his lead in this.

7 Independent natural extension

We now turn to independence assessments, which constitute a symmetrisation of irrelevance assessments. We say that the variables $X_n, n \in N$ are *epistemically independent* when learning the values of any number of them does not influence or change our beliefs about the remaining ones: for any two disjoint subsets I and O of N, X_I is epistemically irrelevant to X_O.

When does a set $\mathscr{D}_N$ of desirable gambles on $\mathscr{X}_N$ capture this type of epistemic independence?

Definition 2. *A coherent set $\mathscr{D}_N$ of desirable gambles on $\mathscr{X}_N$ is called* independent *if*

$$\text{marg}_O(\mathscr{D}_N\rfloor x_I) = \text{marg}_O(\mathscr{D}_N)\quad \textit{for all disjoint } I,O \subseteq N, \textit{ and all } x_I \in \mathscr{X}_I.$$

In this definition, we allow I and O to be empty too, but doing so does not lead to any substantive requirement, because the condition $\text{marg}_O(\mathscr{D}_N\rfloor x_I) = \text{marg}_O(\mathscr{D}_N)$ is trivially satisfied when I or O are empty.

Independent sets have an interesting factorisation property (see Ref. [3] for another paper where factorisation is considered in this somewhat unusual form).

Proposition 14 (Factorisation). *Let $\mathscr{D}_N$ be an independent coherent set of desirable gambles on $\mathscr{X}_N$. Then for all disjoint subsets I and O of N and for all $f \in \mathscr{G}(\mathscr{X}_O)$:*

$$f \in \mathscr{D}_N \Leftrightarrow (\forall g \in \mathscr{G}(\mathscr{X}_I)_{>0}) fg \in \mathscr{D}_N. \tag{11}$$

Proof. Fix arbitrary disjoint subsets I and O of N and any $f \in \mathscr{G}(\mathscr{X}_O)$; we show that Eq. (11) holds. The '⇐' part is trivial. For the '⇒' part, assume that $f \in \mathscr{D}_N$ and consider any $g \in \mathscr{G}(\mathscr{X}_I)_{>0}$. We have to show that $fg \in \mathscr{D}_N$. Since $g = \sum_{x_I \in \mathscr{X}_I} \mathbb{I}_{\{x_I\}} g(x_I)$, we see that $fg = \sum_{x_I \in \mathscr{X}_I} g(x_I)\mathbb{I}_{\{x_I\}} f$. Now since $f \in \text{marg}_O(\mathscr{D}_N)$, we infer from the independence of $\mathscr{D}_N$ and the assumption (i) in Proposition 10 that $f \in \mathscr{D}_N\rfloor x_I$ and therefore $\mathbb{I}_{\{x_I\}} f \in \mathscr{D}_N$ for all $x_I \in \mathscr{X}_I$. We conclude that fg is a positive linear combination of elements $\mathbb{I}_{\{x_I\}} f$ of $\mathscr{D}_N$, and therefore belongs to $\mathscr{D}_N$ by coherence. □

Independence assessments are useful in constructing joint sets of desirable gambles from marginal ones. Suppose we have coherent sets $\mathscr{D}_n$ of desirable gambles on $\mathscr{X}_n$, for each $n \in N$ and an assessment that the variables X_n, $n \in N$ are epistemically independent. Then how can we combine the $\mathscr{D}_n$ and this structural independence assessment into a coherent set of desirable gambles on $\mathscr{X}_N$ in a way that is as conservative as possible? If we call *independent product* of the $\mathscr{D}_n$ any independent $\mathscr{D}_N \in \mathbb{D}(\mathscr{X}_N)$ that marginalises to the $\mathscr{D}_n$ for all $n \in N$, this means we are looking for the smallest such independent product.

Further on, we are going to prove that such a smallest independent product always exists. Before we can do this elegantly, however, we need to do some preparatory work involving particular sets of desirable gambles that can be constructed from the $\mathscr{D}_n$. Consider, as a special case of Eq. (10), for any subset I of N and any $o \in N \setminus I$:

$$\mathscr{A}^{\text{irr}}_{I \to \{o\}} := \operatorname{posi}\left(\left\{\mathbb{I}_{\{x_I\}} g \colon g \in \mathscr{D}_o \text{ and } x_I \in \mathscr{X}_I\right\}\right)$$

It is again easy to see that for all $h \in \mathscr{G}(\mathscr{X}_{I \cup \{o\}})$:

$$h \in \mathscr{A}^{\text{irr}}_{I \to \{o\}} \Leftrightarrow h \neq 0 \text{ and } (\forall x_I \in \mathscr{X}_I) h(x_I, \cdot) \in \mathscr{D}_o \cup \{0\}. \tag{12}$$

We use these sets to construct the following set of desirable gambles on $\mathscr{X}_N$:

$$\otimes_{n \in N} \mathscr{D}_n := \operatorname{posi}\left(\mathscr{G}(\mathscr{X}_N)_{>0} \cup \bigcup_{n \in N} \mathscr{A}^{\text{irr}}_{N \setminus \{n\} \to \{n\}}\right). \tag{13}$$

Observe that, quite trivially, $\mathscr{A}^{\text{irr}}_{\{n\} \setminus \{n\} \to \{n\}} = \mathscr{D}_n$ and therefore $\otimes_{m \in \{n\}} \mathscr{D}_m = \mathscr{D}_n$. We now prove a number of important properties for $\otimes_{n \in N} \mathscr{D}_n$.

Proposition 15 (Coherence). *$\otimes_{n \in N} \mathscr{D}_n$ is a coherent set of desirable gambles on $\mathscr{X}_N$.*

Proof. Let, for ease of notation $\mathscr{A}_N := \bigcup_{n \in N} \mathscr{A}^{\text{irr}}_{N \setminus \{n\} \to \{n\}}$. It follows from Theorem 1 that we have to prove that $\mathscr{A}_N$ avoids non-positivity. So consider any $f \in \operatorname{posi}(\mathscr{A}_N)$, and assume *ex absurdo* that $f \leq 0$. Then there are $\lambda_n \geq 0$ and $f_n \in \mathscr{A}^{\text{irr}}_{N \setminus \{n\} \to \{n\}}$ such that $f = \sum_{n \in N} \lambda_n f_n$ and $\max_{n \in N} \lambda_n > 0$ [recall that the $\mathscr{A}^{\text{irr}}_{N \setminus \{n\} \to \{n\}}$ are convex cones, by Lemma 11]. Fix arbitrary $m \in N$. Let

$$\mathscr{A}^N_m := \left\{f_m(x_{N \setminus \{m\}}, \cdot) \colon x_{N \setminus \{m\}} \in \mathscr{X}_{N \setminus \{m\}}, f_m(x_{N \setminus \{m\}}, \cdot) \neq 0\right\},$$

then it follows from Eq. (12) that $\mathscr{A}^N_m$ is a finite non-empty subset of $\mathscr{D}_m$, so the coherence of $\mathscr{D}_m$, Theorem 1 and Lemma 2 imply that there is some mass function p_m on $\mathscr{X}_m$ with expectation operator E_m such that $(\forall x_m \in \mathscr{X}_m) p_m(x_m) > 0$ and

$$(\forall x_{N \setminus \{m\}} \in \mathscr{X}_{N \setminus \{m\}}) \quad (f_m(x_{N \setminus \{m\}}, \cdot) \neq 0 \Rightarrow E_m(f_m(x_{N \setminus \{m\}}, \cdot)) > 0).$$

So if we define the gamble $g_{N \setminus \{m\}}$ on $\mathscr{X}_{N \setminus \{m\}}$ by letting $g_{N \setminus \{m\}}(x_{N \setminus \{m\}}) := E_m(f_m(x_{N \setminus \{m\}}, \cdot))$ for all $x_{N \setminus \{m\}} \in \mathscr{X}_{N \setminus \{m\}}$, then $g_{N \setminus \{m\}} > 0$.

Since we can do this for all $m \in N$, we can define the mass function p_N on $\mathscr{X}_N$ by letting $p_N(x_N) := \prod_{m \in N} p_m(x_m) > 0$ for all $x_N \in \mathscr{X}_N$. The corresponding expectation operator E_N is of course the product operator of the marginals E_m. But then it follows from the reasoning and assumptions above that $E_N(f) = \sum_{m \in N} \lambda_m E_N(f_m) = \sum_{m \in N} \lambda_m E_N(g_m) > 0$, whereas $f \leq 0$ leads us to conclude that $E_N(f) \leq 0$, a contradiction. □

Lemma 16. *Consider any disjoint subsets I, R of N and any $o \in N \setminus (I \cup R)$. Then $f(x_R, \cdot) \in \mathscr{A}^{\text{irr}}_{I \to \{o\}} \cup \{0\}$ for all $f \in \mathscr{A}^{\text{irr}}_{I \cup R \to \{o\}}$ and all $x_R \in \mathscr{X}_R$.*

Proof. Fix $f \in \mathscr{A}^{\text{irr}}_{I \cup R \to \{o\}}$ and $x_R \in \mathscr{X}_R$ and consider the gamble $g := f(x_R, \cdot)$ on $\mathscr{X}_{I \cup O}$. It follows from the assumptions that for all $x_I \in \mathscr{X}_I$, $g(x_I, \cdot) = f(x_R, x_I, \cdot) \in \mathscr{D}_o \cup \{0\}$, whence indeed $g \in \mathscr{A}^{\text{irr}}_{I \to \{o\}} \cup \{0\}$. □

Proposition 17 (Marginalisation). *Let R be any subset of N, then $\operatorname{marg}_R(\otimes_{n \in N} \mathscr{D}_n) = \otimes_{r \in R} \mathscr{D}_r$.*

Proof. Since we are interpreting gambles on $\mathscr{X}_R$ as special gambles on $\mathscr{X}_N$, it is clear from Eq. (12) that for any $r \in R$, $\mathscr{A}^{\text{irr}}_{R \setminus \{r\} \to \{r\}} \subseteq \mathscr{A}^{\text{irr}}_{N \setminus \{r\} \to \{r\}}$. Eqs. (5) and (13) now tell us that $\operatorname{ext}_N(\otimes_{r \in R} \mathscr{D}_r) \subseteq \otimes_{n \in N} \mathscr{D}_n$. If we invoke Eq. (6), this leads to $\otimes_{r \in R} \mathscr{D}_r = \operatorname{marg}_R(\operatorname{ext}_N(\otimes_{r \in R} \mathscr{D}_r)) \subseteq \operatorname{marg}_R(\otimes_{n \in N} \mathscr{D}_n)$, so we can concentrate on the converse inclusion.

Consider therefore any $f \in \operatorname{marg}_R(\otimes_{n \in N} \mathscr{D}_n) = (\otimes_{n \in N} \mathscr{D}_n) \cap \mathscr{G}(\mathscr{X}_R)$, and assume *ex absurdo* that $f \notin \otimes_{r \in R} \mathscr{D}_r$.

It follows from the coherence of $\otimes_{n \in N} \mathscr{D}_n$ [see Proposition 15] that $f \neq 0$. Since $f \in \otimes_{n \in N} \mathscr{D}_n$, there are $S \subseteq N$, $f_s \in \mathscr{A}^{\text{irr}}_{N \setminus \{s\} \to \{s\}}$, $s \in S$ and $g \in \mathscr{G}(\mathscr{X}_N)$ with $g \geq 0$ such that $f = g + \sum_{s \in S} f_s$. Clearly $S \setminus R \neq \emptyset$, because $S \setminus R = \emptyset$ would imply that, with $x_{N \setminus R}$ any element of $\mathscr{X}_{N \setminus R}$, $f = f(x_{N \setminus R}, \cdot) = g(x_{N \setminus R}, \cdot) + \sum_{s \in S \cap R} f_s(x_{N \setminus R}, \cdot) \in \otimes_{r \in R} \mathscr{D}_r$, since we infer from Lemma 16 that $f_s(x_{N \setminus R}, \cdot) \in \mathscr{A}^{\text{irr}}_{R \setminus \{s\} \to \{s\}} \cup \{0\}$ for all $s \in S \cap R$.

It follows from the coherence of $\otimes_{r \in R} \mathscr{D}_r$ [Proposition 15], $f \notin \otimes_{r \in R} \mathscr{D}_r$ and Lemma 3 that $0 \notin \operatorname{posi}(\{-f\} \cup \otimes_{r \in R} \mathscr{D}_r)$. Let, for ease of notation, $\mathscr{A}^N_{S \cap R}$ be the set

$$\left\{f_s(z_{N \setminus R}, \cdot) \colon s \in S \cap R, z_{N \setminus R} \in \mathscr{X}_{N \setminus R}, f_s(z_{N \setminus R}, \cdot) \neq 0\right\}.$$

Then $\mathscr{A}^N_{S \cap R}$ is clearly a finite subset of $\otimes_{r \in R} \mathscr{D}_r$ [to see this, use a similar argument as above, involving Lemma 16], so we infer from Lemma 2 that there is some mass function p_R on $\mathscr{X}_R$ with associated expectation operator E_R such that

$$\begin{cases} (\forall x_R \in \mathscr{X}_R) p_R(x_R) > 0 \\ (\forall s \in S \cap R)(\forall z_{N \setminus R} \in \mathscr{X}_{N \setminus R}) E_R(f_s(z_{N \setminus R}, \cdot)) \geq 0 \\ E_R(f) < 0. \end{cases}$$

Since $f = f(z_{N \setminus R}, \cdot)$ for any choice of $z_{N \setminus R}$ in $\mathscr{X}_{N \setminus R}$, we see that $f = g(z_{N \setminus R}, \cdot) + \sum_{s \in S \cap R} f_s(z_{N \setminus R}, \cdot) + \sum_{s \in S \setminus R} f_s(z_{N \setminus R}, \cdot)$, whence:

$$\begin{aligned} 0 &> E_R(f) - E_R(g(z_{N \setminus R}, \cdot)) - \sum_{s \in S \cap R} E_R(f_s(z_{N \setminus R}, \cdot)) \\ &= \sum_{s \in S \setminus R} E_R(f_s(z_{N \setminus R}, \cdot)) = \sum_{s \in S \setminus R} \sum_{x_R \in \mathscr{X}_R} p_R(x_R) f_s(z_{N \setminus R}, x_R). \end{aligned}$$

The gambles $f_s(\cdot, x_R)$ on $\mathscr{X}_{N \setminus R}$, with $x_R \in \mathscr{X}_R$ and $s \in S \setminus R$, can clearly not all be zero. The non-zero ones all belong to $\otimes_{s \in N \setminus R} \mathscr{D}_s$, by Lemma 16, so the coherence of the set of desirable gambles $\otimes_{s \in N \setminus R} \mathscr{D}_s$ [Proposition 15] guarantees that their positive linear combination $h := \sum_{s \in S \setminus R} \sum_{x_R \in \mathscr{X}_R} p_R(x_R) f_s(\cdot, x_R)$ also belongs to $\otimes_{s \in N \setminus R} \mathscr{D}_s$. This contradicts $h < 0$. Hence indeed $f \in \otimes_{r \in R} \mathscr{D}_r$. □

Proposition 18 (Conditioning). *$\otimes_{n\in N}\mathscr{D}_n$ is independent: for all disjoint subsets I and O of N, and all $x_I \in \mathscr{X}_I$,*

$$\operatorname{marg}_O(\otimes_{n\in N}\mathscr{D}_n \rfloor x_I) = \operatorname{marg}_O(\otimes_{n\in N}\mathscr{D}_n) = \otimes_{o\in O}\mathscr{D}_o.$$

This could probably be proved indirectly using the 'semi-graphoid' properties of conditional epistemic irrelevance, proved by Moral [7]; it appears we need reverse weak union, reverse decomposition, and contraction. Here we give a direct proof. Proposition 17 can also be seen as a special case of the present result for $I = \emptyset$.

Proof. Fix arbitrary disjoint subsets I and O of N, and arbitrary $x_I \in \mathscr{X}_I$. The second equality follows from Proposition 17, so we concentrate on proving that $\operatorname{marg}_O(\otimes_{n\in N}\mathscr{D}_n \rfloor x_I) = \otimes_{o\in O}\mathscr{D}_o$.

We first show that $\otimes_{o\in O}\mathscr{D}_o \subseteq \otimes_{n\in N}\mathscr{D}_n \rfloor x_I$. Consider any gamble $f \in \otimes_{o\in O}\mathscr{D}_o$, then we have to show that $\mathbb{I}_{\{x_I\}} f \in \otimes_{n\in N}\mathscr{D}_n$. By assumption, there are non-negative reals λ_o and μ, gambles $f_o \in \mathscr{A}^{\text{irr}}_{O\setminus\{o\}\to\{o\}}$ for all $o \in O$ and $g \in \mathscr{G}(\mathscr{X}_O)_{>0}$ such that $f = \mu g + \sum_{o\in O}\lambda_o f_o$ and $\max\{\mu, \max_{o\in O}\lambda_o\} > 0$. Fix $o \in O$ and let $f'_o := \mathbb{I}_{\{x_I\}} f_o \in \mathscr{G}(\mathscr{X}_N)$. Then it follows from the definition of $\mathscr{A}^{\text{irr}}_{O\setminus\{o\}\to\{o\}}$ that $f'_o(z_{N\setminus\{o\}}, \cdot) = \mathbb{I}_{\{x_I\}}(z_I) f_o(z_{O\setminus\{o\}}, \cdot) \in \mathscr{D}_o \cup \{0\}$ for all $z_{N\setminus\{o\}} \in \mathscr{X}_{N\setminus\{o\}}$. Since $f'_o \neq 0$, the definition of $\mathscr{A}^{\text{irr}}_{N\setminus\{o\}\to\{o\}}$ tells us that $f'_o \in \mathscr{A}^{\text{irr}}_{N\setminus\{o\}\to\{o\}}$. Similarly, if we let $g' := \mathbb{I}_{\{x_I\}} g \in \mathscr{G}(\mathscr{X}_N)$, then $g' > 0$. So it follows from Eq. (13) that indeed $\mathbb{I}_{\{x_I\}} f = \mu g' + \sum_{o\in O}\lambda_o f'_o \in \otimes_{n\in N}\mathscr{D}_n$.

We now turn to the converse inclusion $\otimes_{n\in N}\mathscr{D}_n \rfloor x_I \subseteq \otimes_{o\in O}\mathscr{D}_o$. Consider any gamble $f \in \mathscr{G}(\mathscr{X}_O)$ such that $\mathbb{I}_{\{x_I\}} f$ belongs to $\otimes_{n\in N}\mathscr{D}_n$ and assume *ex absurdo* that $f \notin \otimes_{o\in O}\mathscr{D}_o$. Let, for the sake of notational simplicity, $C := N \setminus (I \cup O)$.

It follows from the coherence of $\otimes_{n\in N}\mathscr{D}_n$ [Proposition 15] that $f \neq 0$. Since $\mathbb{I}_{\{x_I\}} f \in \otimes_{n\in N}\mathscr{D}_n$, there are $S \subseteq N$, $f_s \in \mathscr{A}^{\text{irr}}_{N\setminus\{s\}\to\{s\}}$, $s \in S$ and $g \in \mathscr{G}(\mathscr{X}_N)$ with $g \geq 0$ such that $\mathbb{I}_{\{x_I\}} f = g + \sum_{s\in S} f_s$. Clearly $S \setminus O \neq \emptyset$, because $S \setminus O = \emptyset$ would imply that, with x_C any element of $\mathscr{X}_C$, $f = g(x_I, x_C, \cdot) + \sum_{s\in S\cap O} f_s(x_I, x_C, \cdot) \in \otimes_{o\in O}\mathscr{D}_o$, because $f_s(x_I, x_C, \cdot) \in \mathscr{A}^{\text{irr}}_{O\setminus\{s\}\to\{s\}}$ for all $s \in S \cap O$ by Lemma 16.

It follows from the coherence of $\otimes_{o\in O}\mathscr{D}_o$ [Proposition 15], $f \notin \otimes_{o\in O}\mathscr{D}_o$ and Lemma 3 that $0 \notin \operatorname{posi}(\{-f\} \cup \otimes_{o\in O}\mathscr{D}_o)$. The set

$$\mathscr{A}^N_{S\cap O} := \{f_s(x_I, z_C, \cdot) \colon s \in S \cap O, z_C \in \mathscr{X}_C, f_s(x_I, z_C, \cdot) \neq 0\}$$

is clearly a finite subset of $\otimes_{o\in O}\mathscr{D}_o$ [use Lemma 16 again], so we infer from Lemma 2 that there is some mass function p_O on $\mathscr{X}_O$ with associated expectation operator E_O such that

$$\begin{cases} (\forall x_O \in \mathscr{X}_O) p_O(x_O) > 0 \\ (\forall s \in S \cap O)(\forall z_C \in \mathscr{X}_C) E_O(f_s(x_I, z_C, \cdot)) \geq 0 \\ E_O(f) < 0. \end{cases}$$

Since $f = g(x_I, z_C, \cdot) + \sum_{s\in S\cap O} f_s(x_I, z_C, \cdot) + \sum_{s\in S\setminus O} f_s(x_I, z_C, \cdot)$ for any choice of $z_C \in \mathscr{X}_C$, we see that:

$$\begin{aligned} 0 &> E_O(f) - E_O(g(x_I, z_C, \cdot)) - \sum_{s\in S\cap O} E_O(f_s(x_I, z_C, \cdot)) \\ &= \sum_{s\in S\setminus O} E_O(f_s(x_I, z_C, \cdot)) = \sum_{s\in S\setminus O}\sum_{x_O\in\mathscr{X}_O} p_O(x_O) f_s(x_I, z_C, x_O). \end{aligned}$$

Similarly, for any $z_C \in \mathscr{X}_C$ and any $z_I \in \mathscr{X}_I \setminus \{x_I\}$ we infer from $0 = g(z_I, z_C, \cdot) + \sum_{s\in S\cap O} f_s(z_I, z_C, \cdot) + \sum_{s\in S\setminus O} f_s(z_I, z_C, \cdot)$ that:

$$\begin{aligned} 0 &\geq -E_O(g(z_I, z_C, \cdot)) - \sum_{s\in S\cap O} E_O(f_s(z_I, z_C, \cdot)) \\ &= \sum_{s\in S\setminus O} E_O(f_s(z_I, z_C, \cdot)) = \sum_{s\in S\setminus O}\sum_{x_O\in\mathscr{X}_O} p_O(x_O) f_s(z_I, z_C, x_O). \end{aligned}$$

Hence $h := \sum_{s\in S\setminus O}\sum_{x_O\in\mathscr{X}_O} p_O(x_O) f_s(\cdot, \cdot, x_O) < 0$. The gambles $f_s(\cdot, \cdot, x_O)$ on $\mathscr{X}_{I\cup C}$, with $x_O \in \mathscr{X}_O$ and $s \in S \setminus O$, can clearly not all be zero. The non-zero ones all belong to $\otimes_{s\in I\cup C}\mathscr{D}_s$, by Lemma 16. But then the coherence of the set of desirable gambles $\otimes_{s\in I\cup C}\mathscr{D}_s$ [Proposition 15] guarantees that their positive linear combination h is an element of $\otimes_{c\in C}\mathscr{D}_c$ for which $h < 0$, a contradiction. Hence indeed $f \in \otimes_{o\in O}\mathscr{D}_o$. ☐

Theorem 19 (Independent natural extension). *$\otimes_{n\in N}\mathscr{D}_n$ is the smallest coherent set of desirable gambles on $\mathscr{X}_N$ that is an independent product of the coherent sets $\mathscr{D}_n$ of desirable gambles on $\mathscr{X}_n$, $n \in N$.*

We call $\otimes_{n\in N}\mathscr{D}_n$ the *independent natural extension* of the marginals $\mathscr{D}_n$.

Proof. It follows from Propositions 15, 17 and 18 that $\otimes_{n\in N}\mathscr{D}_n$ is an independent product $\mathscr{D}_N$ of the $\mathscr{D}_n$. To prove that it is the smallest one, consider any independent product $\mathscr{D}_N$ of the $\mathscr{D}_n$. Fix $n \in N$. If we consider any $x_{N\setminus\{n\}} \in \mathscr{X}_{N\setminus\{n\}}$, then $\operatorname{marg}_n(\mathscr{D}_N \rfloor x_{N\setminus\{n\}}) = \mathscr{D}_n$, by assumption. If we therefore consider any $g \in \mathscr{D}_n$, this in turn implies that $g \in \mathscr{D}_N \rfloor x_{N\setminus\{n\}}$, and therefore $\mathbb{I}_{\{x_{N\setminus\{n\}}\}} g \in \mathscr{D}_N$, by Eq. (8). So we infer by coherence that $\mathscr{A}^{\text{irr}}_{N\setminus\{n\}\to\{n\}} \subseteq \mathscr{D}_N$, and therefore also that $\otimes_{n\in N}\mathscr{D}_n \subseteq \mathscr{D}_N$. ☐

Theorem 20 (Associativity). *Let N_1, N_2 be disjoint non-empty index sets, and let $\mathscr{D}_{n_k} \in \mathbb{D}(\mathscr{X}_{n_k})$, $n_k \in N_k$, $k = 1, 2$. Then $\otimes_{n\in N_1\cup N_2}\mathscr{D}_n = (\otimes_{n_1\in N_1}\mathscr{D}_{n_1}) \otimes (\otimes_{n_2\in N_2}\mathscr{D}_{n_2})$.*

Proof. Consider, for ease of notation, $\mathscr{D}_{N_1} := \otimes_{n_1\in N_1}\mathscr{D}_{n_1}$ and $\mathscr{D}_{N_2} := \otimes_{n_2\in N_2}\mathscr{D}_{n_2}$. We have to prove that $\mathscr{D}_{N_1} \otimes \mathscr{D}_{N_2} = \otimes_{n\in N_1\cup N_2}\mathscr{D}_n$.

We first prove that $\mathscr{D}_{N_1} \otimes \mathscr{D}_{N_2} \subseteq \otimes_{n\in N_1\cup N_2}\mathscr{D}_n$. Fix any gamble $h \in \mathscr{A}^{\text{irr}}_{\{N_1\}\to\{N_2\}}$ and any $x_{N_1} \in \mathscr{X}_{N_1}$, so $h(x_{N_1}, \cdot) \in \mathscr{D}_{N_2} \cup \{0\}$ by Eq. (12). It follows from Eq. (13) that there are gambles $h^{n_2}_{x_{N_1}} \in \mathscr{A}^{\text{irr}}_{N_2\setminus\{n_2\}\to\{n_2\}} \cup \{0\}$ for all $n_2 \in N_2$ such that $h(x_{N_1}, \cdot) \geq \sum_{n_2\in N_2} h^{n_2}_{x_{N_1}}$. Define, for any $n_2 \in N_2$, the gamble g_{n_2} on $\mathscr{X}_N$ by letting $g_{n_2}(x_{N\setminus\{n_2\}}, \cdot) := h^{n_2}_{x_{N_1}}(x_{N_2\setminus\{n_2\}}, \cdot)$ for all $x_N \in \mathscr{X}_N$. Then it follows from Eq. (12) that $g_{n_2}(x_{N\setminus\{n_2\}}, \cdot) \in \mathscr{D}_{n_2} \cup \{0\}$ for all $x_N \in \mathscr{X}_N$, and therefore $g_{n_2} \in \mathscr{A}^{\text{irr}}_{N\setminus\{n_2\}\to\{n_2\}} \cup \{0\}$. Moreover,

$$\begin{aligned} h &= \sum_{x_{N_1}\in\mathscr{X}_{N_1}} \mathbb{I}_{\{x_{N_1}\}} h(x_{N_1}, \cdot) \geq \sum_{x_{N_1}\in\mathscr{X}_{N_1}} \mathbb{I}_{\{x_{N_1}\}} \sum_{n_2\in N_2} h^{n_2}_{x_{N_1}} \\ &= \sum_{n_2\in N_2}\sum_{x_{N_1}\in\mathscr{X}_{N_1}} \mathbb{I}_{\{x_{N_1}\}} h^{n_2}_{x_{N_1}} = \sum_{n_2\in N_2} g_{n_2}, \end{aligned}$$

Since clearly $h \neq 0$, we infer from Eq. (13) that $h \in \otimes_{n\in N_1\cup N_2}\mathscr{D}_n$. We conclude that $\mathscr{A}^{\text{irr}}_{\{N_1\}\to\{N_2\}} \subseteq \otimes_{n\in N_1\cup N_2}\mathscr{D}_n$. Similarly, we can prove the inclusion $\mathscr{A}^{\text{irr}}_{\{N_2\}\to\{N_1\}} \subseteq \otimes_{n\in N_1\cup N_2}\mathscr{D}_n$, and therefore also $\mathscr{D}_{N_1} \otimes \mathscr{D}_{N_2} \subseteq \otimes_{n\in N_1\cup N_2}\mathscr{D}_n$, again by Eq. (13).

To conclude, we turn to the converse inclusion $\otimes_{n\in N_1\cup N_2}\mathscr{D}_n \subseteq \mathscr{D}_{N_1} \otimes \mathscr{D}_{N_2}$. Consider any gamble $h \in \otimes_{n\in N_1\cup N_2}\mathscr{D}_n$, then by Eq. (13) there are $h_n \in \mathscr{A}^{\text{irr}}_{N_1\cup N_2\setminus\{n\}\to\{n\}} \cup \{0\}$, $n \in N_1 \cup N_2$, such that $h \geq h_1 + h_2$, where we let $h_1 := \sum_{n_1 \in N_1} h_{n_1}$ and $h_2 := \sum_{n_2 \in N_2} h_{n_2}$. Fix any $x_{N_1} \in \mathscr{X}_{N_1}$. For any $n_2 \in N_2$, we infer that $h_{n_2}(x_{N_1}, \cdot) \in \mathscr{A}^{\mathrm{irr}}_{N_2 \setminus \{n_2\} \to \{n_2\}} \cup \{0\}$ from $h_{n_2} \in \mathscr{A}^{\mathrm{irr}}_{N_1 \cup N_2 \setminus \{n_2\} \to \{n_2\}} \cup \{0\}$ by Lemma 16. Hence $h_2(x_{N_1}, \cdot) \in \mathscr{D}_{N_2} \cup \{0\}$ by Eq. (13), and therefore $h_2 \in \mathscr{A}^{\mathrm{irr}}_{\{N_1\} \to \{N_2\}} \cup \{0\}$ by Eq. (12). Similarly, $h_1 \in \mathscr{A}^{\mathrm{irr}}_{\{N_2\} \to \{N_1\}} \cup \{0\}$, and therefore $h \in \mathscr{D}_{N_1} \otimes \mathscr{D}_{N_2}$ by Eq. (13), since clearly $h \neq 0$. □

To conclude this section, we establish a connection between independent natural extension for sets of desirable gambles and the eponymous notion for coherent lower previsions studied in detail in Ref. [3]. Given coherent lower previsions $\underline{P}_n$ on $\mathscr{G}(\mathscr{X}_n)$, $n \in N$, their *independent natural extension* is the coherent lower prevision given by

$$\underline{E}_N(f) := \sup_{\substack{h_n \in \mathscr{G}(\mathscr{X}_N) \\ n \in N}} \min_{z_N \in \mathscr{X}_N} \left[f(z_N) - \sum_{n \in N} [h_n(z_N) - \underline{P}_n(h_n(\cdot, z_{N \setminus \{n\}}))] \right] \tag{14}$$

for all gambles f on $\mathscr{X}_N$. It is the point-wise smallest (most conservative) joint lower prevision that is jointly coherent with the marginals $\underline{P}_n$ given an assessment of epistemic independence of the variables X_n, $n \in N$.

Theorem 21. *Let $\mathscr{D}_n$ be coherent sets of desirable gambles on $\mathscr{X}_n$ for $n \in N$, and let $\otimes_{n \in N} \mathscr{D}_n$ be their independent natural extension. Consider the coherent lower previsions $\underline{P}_n$ on $\mathscr{G}(\mathscr{X}_n)$ given by $\underline{P}_n(f_n) := \sup \{\mu \in \mathbb{R} \colon f_n - \mu \in \mathscr{D}_n\}$ for all $f_n \in \mathscr{G}(\mathscr{X}_n)$. Then the independent natural extension $\underline{E}_N$ of the marginal lower previsions $\underline{P}_n$, $n \in N$ satisfies*

$$\underline{E}_N(f) = \sup \{\mu \in \mathbb{R} \colon f - \mu \in \otimes_{n \in N} \mathscr{D}_n\}$$

for all gambles f on $\mathscr{X}_N$.

Proof. Fix any gamble f in $\mathscr{G}(\mathscr{X}_N)$. First, consider any real number $\mu < \underline{E}_N(f)$, then it follows from Eq. (14) that there are $\delta > 0$ and $h_n \in \mathscr{G}(\mathscr{X}_N)$, $n \in N$, such that $f - \mu \geq \sum_{n \in N} g_n$, where we defined the gambles g_n on $\mathscr{X}_N$ by $g_n(z_N) := h_n(z_N) - \underline{P}_n(h_n(z_{N \setminus \{n\}}, \cdot)) + \delta$ for all $z_N \in \mathscr{X}_N$. It follows from the definition of $\underline{P}_n$ that $g_n(z_{N \setminus \{n\}}, \cdot) = h_n(z_{N \setminus \{n\}}, \cdot) - \underline{P}_n(h_n(z_{N \setminus \{n\}}, \cdot)) + \delta \in \mathscr{D}_n$ for all $z_{N \setminus \{n\}} \in \mathscr{X}_{N \setminus \{n\}}$. Since clearly $g_n \neq 0$, Eq. (12) then tells us that $g_n \in \mathscr{A}^{\mathrm{irr}}_{N \setminus \{n\} \to \{n\}}$, and we infer from Eq. (13) that $\sum_{n \in N} g_n \in \otimes_{n \in N} \mathscr{D}_n$, and therefore also $f - \mu \in \otimes_{n \in N} \mathscr{D}_n$. This guarantees that $\underline{E}_N(f) \leq \sup \{\mu \in \mathbb{R} \colon f - \mu \in \otimes_{n \in N} \mathscr{D}_n\}$.

To prove the converse inequality, consider any real μ such that $f - \mu \in \otimes_{n \in N} \mathscr{D}_n$. We infer using Eq. (13) that there are gambles $h_n \in \mathscr{A}^{\mathrm{irr}}_{N \setminus \{n\} \to \{n\}} \cup \{0\}$, $n \in N$, such that $f - \mu \geq \sum_{n \in N} h_n$. For all $n \in N$ and $z_{N \setminus \{n\}} \in \mathscr{X}_{N \setminus \{n\}}$, it follows from Eq. (12) that $h_n(z_{N \setminus \{n\}}, \cdot) \in \mathscr{D}_n \cup \{0\}$, whence $\underline{P}_n(h_n(z_{N \setminus \{n\}}, \cdot)) \geq 0$. This leads to $\sum_{n \in N} [h_n(z_N) - \underline{P}_n(h_n(z_{N \setminus \{n\}}, \cdot))] \leq \sum_{n \in N} h_n(z_N) \leq f(z_N) - \mu$. We then infer from Eq. (14) that $\underline{E}_N(f) \geq \mu$ and so we find that indeed also $\underline{E}_N(f) \geq \sup \{\mu \in \mathbb{R} \colon f - \mu \in \otimes_{n \in N} \mathscr{D}_n\}$. □

8 Maximal sets of desirable gambles and strong products

The following result was (essentially) proved in Ref. [1].

Proposition 22. *Let $\mathscr{M}_N \in \mathbb{M}(\mathscr{X}_N)$, and consider any disjoint subsets I and O of N. Then $\mathrm{marg}_O(\mathscr{M}_N \rfloor x_I) \in \mathbb{M}(\mathscr{X}_O)$ for all $x_I \in \mathscr{X}_I$.*

Now consider the case where we have coherent marginal sets of desirable gambles $\mathscr{D}_n$ for all $n \in N$. We define their *strong product* $\boxtimes_{n \in N} \mathscr{D}_n$ as the set of desirable gambles on the product space $\mathscr{X}_N$ given by:

$$\boxtimes_{n \in N} \mathscr{D}_n := \bigcap \{\otimes_{n \in N} \mathscr{M}_n \colon \mathscr{M}_n \in m(\mathscr{D}_n), n \in N\}$$

Observe that for maximal sets $\mathscr{M}_n \in \mathbb{M}(\mathscr{X}_n)$, $n \in N$ the strong product and the independent natural extension coincide: $\boxtimes_{n \in N} \mathscr{M}_n = \otimes_{n \in N} \mathscr{M}_n$.

The marginalisation properties of the strong product follow from those of the independent natural extension.

Proposition 23 (Marginalisation). *Consider coherent sets of desirable gambles $\mathscr{D}_n$ for all $n \in N$. Let R be any subset of N, then $\mathrm{marg}_R(\boxtimes_{n \in N} \mathscr{D}_n) = \boxtimes_{r \in R} \mathscr{D}_r$.*

Proof. Consider any $f \in \mathscr{G}(\mathscr{X}_R)$ and observe the following chain of equivalences:

$$\begin{aligned} f \in \boxtimes_{n \in N} \mathscr{D}_n &\Leftrightarrow (\forall \mathscr{M}_n \in m(\mathscr{D}_n), n \in N) f \in \otimes_{n \in N} \mathscr{M}_n \\ &\Leftrightarrow (\forall \mathscr{M}_n \in m(\mathscr{D}_n), n \in N) f \in \otimes_{r \in R} \mathscr{M}_r \\ &\Leftrightarrow (\forall \mathscr{M}_r \in m(\mathscr{D}_r), r \in R) f \in \otimes_{r \in R} \mathscr{M}_r \\ &\Leftrightarrow f \in \boxtimes_{r \in R} \mathscr{D}_r, \end{aligned}$$

where the second equivalence follows from Proposition 17. □

As we have come to expect from our treatment of the independent natural extension, the proof of the following independence property is very similar to that of the marginalisation property.

Proposition 24. *Consider coherent sets of desirable gambles $\mathscr{D}_n$ for all $n \in N$. Then their strong product $\boxtimes_{n \in N} \mathscr{D}_n$ is an independent product of these marginals.*

Proof. Consider any disjoint subsets I and O of N, and any $x_I \in \mathscr{X}_I$, then it suffices to prove that, also using Proposition 23, $\mathrm{marg}_O(\boxtimes_{n \in N} \mathscr{D}_n \rfloor x_I) = \boxtimes_{o \in O} \mathscr{D}_o$. So consider any gamble f on $\mathscr{X}_O$ and observe the following chain of equivalences:

$$\begin{aligned} f \in \boxtimes_{n \in N} \mathscr{D}_n \rfloor x_I &\Leftrightarrow \mathbb{I}_{\{x_I\}} f \in \boxtimes_{n \in N} \mathscr{D}_n \\ &\Leftrightarrow (\forall \mathscr{M}_n \in m(\mathscr{D}_n), n \in N) \mathbb{I}_{\{x_I\}} f \in \otimes_{n \in N} \mathscr{M}_n \\ &\Leftrightarrow (\forall \mathscr{M}_n \in m(\mathscr{D}_n), n \in N) f \in \otimes_{o \in O} \mathscr{M}_o \\ &\Leftrightarrow (\forall \mathscr{M}_o \in m(\mathscr{D}_o), o \in O) f \in \otimes_{o \in O} \mathscr{M}_o \\ &\Leftrightarrow f \in \boxtimes_{o \in O} \mathscr{D}_o, \end{aligned}$$

where the third equivalence follows from Proposition 18. □

It is still an open problem at this point whether, like the natural extension, the strong product is associative.

To conclude this section, we establish a connection between the strong product of sets of desirable gambles and the eponymous notion for coherent lower previsions, studied in Ref. [3]. Given coherent lower previsions $\underline{P}_n$ on $\mathscr{G}(\mathscr{X}_n)$, $n \in N$, their *strong product* is defined by

$$\underline{S}_N(f) := \inf\{\times_{n\in N} P_n(f) \colon (\forall n \in N) P_n \in \mathscr{M}(\underline{P}_n)\}$$

for all gambles f on $\mathscr{X}_N$. If we start from linear previsions P_n on $\mathscr{G}(\mathscr{X}_n)$, their strong product corresponds to their linear product $\times_{n\in N} P_n$, and it coincides also with their independent natural extension E_N. If we begin with coherent lower previsions $\underline{P}_n$ on $\mathscr{G}(\mathscr{X}_n)$, their strong product $\underline{S}_N$ is the lower envelope of the set of strong products determined by the dominating linear previsions.

Theorem 25. *Let $\mathscr{D}_n$ be coherent sets of desirable gambles in $\mathscr{G}(\mathscr{X}_n)$ for all $n \in N$, and let $\boxtimes_{n\in N}\mathscr{D}_n$ be their strong product. Consider the coherent lower previsions $\underline{P}_n$ on $\mathscr{G}(\mathscr{X}_n)$ given by $\underline{P}_n(f) := \sup\{\mu \in \mathbb{R} \colon f - \mu \in \mathscr{D}_n\}$. Then the strong product $\underline{S}_N$ of the marginal lower previsions $\underline{P}_n$, $n \in N$ satisfies $\underline{S}_N(f) = \sup\{\mu \in \mathbb{R} \colon f - \mu \in \boxtimes_{n\in N}\mathscr{D}_n\}$.*

Proof. Assume first of all that $\mathscr{D}_n$ is a maximal set of desirable gambles for all n in N. Then it follows from Theorem 3.8.3 in Ref. [8] that $\underline{P}_n$ is a linear prevision, which we denote by P_n, for all $n \in N$. The strong product of the linear previsions P_n, $n \in N$ coincides with their linear independent product $\times_{n\in N} P_n$, which is also their independent natural extension, by Proposition 10 in Ref. [3]. Since we have proved in Theorem 21 that this is the coherent lower prevision associated with $\otimes_{n\in N}\mathscr{D}_n = \boxtimes_{n\in N}\mathscr{D}_n$, we conclude that the strong product $\boxtimes_{n\in N}\mathscr{D}_n$ is associated with the strong product of the linear previsions P_n.

Next, fix any gamble f on $\mathscr{X}_N$. Consider any real number $\mu < \underline{S}_N(f)$. For any $n \in N$, consider any maximal set $\mathscr{M}_n \in m(\mathscr{D}_n)$, and the associated linear prevision P_n, then clearly $P_n \in \mathscr{M}(\underline{P}_n)$. Hence $\times_{n\in N} P_n(f) \geq \underline{S}_N(f) > \mu$, and we infer from the arguments above that then necessarily $f - \mu \in \otimes_{n\in N}\mathscr{M}_n$. Hence $f - \mu \in \boxtimes_{n\in N}\mathscr{D}_n$. This leads to the conclusion that $\underline{S}_N(f) \leq \sup\{\mu \in \mathbb{R} \colon f - \mu \in \boxtimes_{n\in N}\mathscr{D}_n\}$.

Conversely, consider any real μ such that $f - \mu \in \boxtimes_{n\in N}\mathscr{D}_n$. Consider arbitrary $P_n \in \mathscr{M}(\underline{P}_n)$, $n \in N$, then there are maximal sets $\mathscr{M}_n \in m(\mathscr{D}_n)$ inducing them: indeed, the set of strictly desirable gambles $\mathscr{D}_n'$ that induces P_n, given by Eq. (2), is coherent by Theorem 3.8.1 in Ref. [8]; Theorem 5 implies that there is some maximal set $\mathscr{M}_n \in m(\mathscr{D}_n') \supseteq m(\mathscr{D}_n)$, and now Theorem 3.8.3 in Ref. [8] implies that $\mathscr{D}_n'$ and $\mathscr{M}_n$ induce the same P_n by means of Eq. (1). But then $f - \mu \in \otimes_{n\in N}\mathscr{M}_n$, and therefore $\times_{n\in N} P_n(f) \geq \mu$, using the argumentation above. Hence $\underline{S}_N(f) \geq \mu$, and therefore also $\underline{S}_N(f) \geq \sup\{\mu \in \mathbb{R} \colon f - \mu \in \boxtimes_{n\in N}\mathscr{D}_n\}$. □

Together with Theorem 21 and the fact that the strong product of lower previsions may strictly dominate their independent natural extension [see Example 9.3.4 in Ref. [8]], this shows that the strong product of marginal sets of desirable gambles may strictly include their independent natural extension.

9 Conditional irrelevance and independence

We turn to conditional irrelevance judgements. Next to the variables X_N in $\mathscr{X}_N$, we now also consider another variable Y assuming values in a finite set $\mathscr{Y}$.

Consider two disjoint subsets I and O of N. We say that X_I is *epistemically irrelevant* to X_O when, conditional on Y, learning the value of X_I does not influence or change our beliefs about X_O. In order for a set $\mathscr{D}$ of desirable gambles on $\mathscr{X}_N \times \mathscr{Y}$ to capture this type of conditional epistemic irrelevance, we should require that:

$$\mathrm{marg}_O(\mathscr{D}\rfloor x_I, y) = \mathrm{marg}_O(\mathscr{D}\rfloor y) \; \forall x_I \in \mathscr{X}_I, y \in \mathscr{Y}.$$

As before, for technical reasons we also allow I and O to be empty. It is clear from the definition above that the 'variable' $X_\emptyset$, about whose constant value we are certain, is conditionally epistemically irrelevant to any variable X_O. Similarly, we see that any variable X_I is conditionally epistemically irrelevant to the 'variable' $X_\emptyset$. This seems to be in accordance with intuition.

Also, if $\mathscr{Y}$ is a singleton, then there is no uncertainty about Y and conditioning on Y amounts to not conditioning at all: epistemic irrelevance can be seen as a special case of conditional epistemic irrelevance. We now want to argue that, conversely, there is a very specific and definite way in which conditional epistemic irrelevance statements can be reduced to simple epistemic irrelevance statements. The crucial results that allow us to establish this, are the following conceptually very simple theorem and its corollary.

Theorem 26 (Sequential updating). *Consider any subset R of N, and any coherent set $\mathscr{D}$ of desirable gambles on $\mathscr{X}_N \times \mathscr{Y}$. Then*

$$(\mathscr{D}\rfloor y)\rfloor x_R = (\mathscr{D}\rfloor x_R)\rfloor y = \mathscr{D}\rfloor x_R, y \quad \text{for all } x_R \in \mathscr{X}_R \text{ and } y \in \mathscr{Y}. \tag{15}$$

Proof. Fix any x_R in $\mathscr{X}_R$ and any $y \in \mathscr{Y}$. Clearly, all three sets in Eq. (15) are subsets of $\mathscr{G}(\mathscr{X}_{N\setminus R})$. So take any gamble f on $\mathscr{X}_{N\setminus R}$, and consider the following chains of equivalences:

$$\mathbb{I}_{\{y\}}\mathbb{I}_{\{x_R\}} f \in \mathscr{D} \Leftrightarrow \mathbb{I}_{\{x_R\}} f \in \mathscr{D}\rfloor y \Leftrightarrow f \in (\mathscr{D}\rfloor y)\rfloor x_R$$
$$\mathbb{I}_{\{y\}}\mathbb{I}_{\{x_R\}} f \in \mathscr{D} \Leftrightarrow \mathbb{I}_{\{y\}} f \in \mathscr{D}\rfloor x_R \Leftrightarrow f \in (\mathscr{D}\rfloor x_R)\rfloor y$$
$$\mathbb{I}_{\{y\}}\mathbb{I}_{\{x_R\}} f \in \mathscr{D} \Leftrightarrow f \in \mathscr{D}\rfloor x_R, y.$$
□

Corollary 27 (Reduction). *Consider any disjoint subsets I and O of N, and any coherent set $\mathscr{D}$ of desirable gambles on $\mathscr{X}_N \times \mathscr{Y}$. Then the following statements are equivalent:*

(i) $\mathrm{marg}_O(\mathscr{D}\rfloor x_I, y) = \mathrm{marg}_O(\mathscr{D}\rfloor y)$ *for all $x_I \in \mathscr{X}_I$ and all $y \in \mathscr{Y}$;*
(ii) $\mathrm{marg}_O((\mathscr{D}\rfloor y)\rfloor x_I) = \mathrm{marg}_O(\mathscr{D}\rfloor y)$ *for all $x_I \in \mathscr{X}_I$ and all $y \in \mathscr{Y}$.*

This tells us that *a model* $\mathscr{D}$ *about* (X_N, Y) *captures epistemic irrelevance of* X_I *to* X_O*, conditional on* Y *if and only if for each possible value* $y \in \mathscr{Y}$ *of* Y*, the model* $\mathscr{D}\rfloor y$ *about* X_N *captures epistemic irrelevance of* X_I *to* X_O*.*

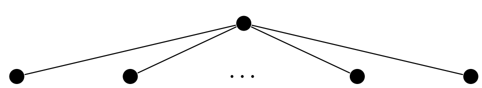
Now suppose we have marginal conditional models $\mathscr{D}_n\rfloor Y$ on $\mathscr{X}_n$, $n \in N$. The notation $\mathscr{D}_n\rfloor Y$ is a concise way of representing the family of conditional models $\mathscr{D}_n\rfloor y$, $y \in \mathscr{Y}$. Then if we combine Corollary 27 and Theorem 19, we see that the smallest conditionally independent product $\mathscr{D}\rfloor Y$ of these marginal models $\mathscr{D}_n\rfloor Y$ is given by $\otimes_{n\in N}(\mathscr{D}_n\rfloor Y)$, meaning that for each $y \in \mathscr{Y}$, $\mathscr{D}\rfloor y = \otimes_{n\in N}(\mathscr{D}_n\rfloor y)$.

10 Conclusions

Sets of desirable gambles are more informative than coherent lower previsions, and they are helpful in avoiding problems involving zero probabilities. They have been overlooked for much of the development of the theory, and it is only in the last five or six years that more effort is being devoted to bringing this simplifying and unifying notion to the fore.

Our results here show that we can model assessments of epistemic independence easily using sets of desirable gambles, and that we can derive from them existing results for lower previsions.

They also indicate that constructing global joint models (i.e. coherent sets of desirable gambles) from local ones is something that can be easily and efficiently done for the following types of simple credal networks:



They may therefore open up the way towards finding efficient algorithms for inference in credal *trees* under epistemic irrelevance using sets of desirable gambles as uncertainty models, building on the ideas proposed in Ref. [2]. We expect that generalising those algorithms towards more general credal networks (polytrees, ...) will be more difficult, and will have to rely heavily on the pioneering work of Moral [7] on graphoid properties for epistemic irrelevance.

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