A study on updating belief functions for parameter uncertainty representation in Nuclear Probabilistic Risk Assessment

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Abstract

Probabilistic Risk Assessments (PRA) are used to achieve a safe design and operation of Nuclear Power Plants. The impact of uncertainties which may affect PRA results must thus be taken into account in the decision making process. These uncertainties due to the lack of data have been recently seen as mainly epistemic ones and it has been recommended to characterize them by the belief functions of Dempster-Shafer Theory rather than a presumed single probability distribution. The current construction of these functions is based on the data provided by PRA data handbooks using traditional statistical tools like Maximum Likelihood Estimation (MLE). However, this approach is only appropriate when data coming from the operating feedback observations are sufficiently large as required in the MLE approach. Furthermore, when wishing to incorporate other sources of information, such as expert's opinions, the pooling data of MLE has limits to account for these kinds of information. Therefore, in order to overcome this problem, two alternative perspectives based on the Dempster's rule of combination and the Generalized Bayesian Theorem for constructing and updating the belief functions in a more effective way will be presented in this paper. These two approaches will be studied for the use in the context of PRA. The comparison of these two approaches with the current method is carried out through a practical example. Some conclusions about the application of these approaches will be drawn.

Keywords. Parameter uncertainty, belief functions, generalized Bayesian theorem, nuclear risk assessment.

1 Introduction

Probabilistic Risk Assessment (PRA) [10] is a methodology which provides a quantitative assessment of the risk of accidents at Nuclear Power Plants (NPP). It involves the development of models that delineate the response of systems and of operators to initiating events that could lead to core damage or a release of radioactivity to the environment. The evaluation of the frequency of such an accident relies on the assessment of the failure probability of systems by means of event/fault trees. In PRA, parametric statistical models are used to characterize the random occurrence of accidents at nuclear power plants [2][10]. Some usual parametric models like Poisson model, exponential model…are used for this purpose. The parameters associated to these models in PRA are reliability parameters such as the failure rates of individual components or the probability of failure on demand and so on. The values of these parameters are generally unknown and estimated with statistical tools. These estimated values are therefore subjected to uncertainty due to insufficient feedback data which can impact the decision making process. As a consequence, the results in the nuclear PRA context for decision making need to take into account these uncertainties.

In the traditional PRA practice of uncertainty analysis, the epistemic parameter uncertainty is generally represented by a presumed probability distribution, such as the log-normal distribution which is viewed as the subjective interpretation of probability (i.e. degree of belief) for the possible values of the parameter. Nevertheless, the choice of this distribution which is made for some practical reasons has been shown to be questionable because it could have major impacts on the final results of decision making [20]. Recently, a general framework of parameter uncertainty quantification within the Dempster-Shafer Theory (DST) framework has been proposed in the nuclear PRA context [20][21]. In this framework, parameter uncertainty is no longer characterized by an assumed probability distribution but by belief and plausibility functions which represent the current state of knowledge about the possible values of the parameter. The approach proposed in [20] for the construction of these belief functions is based on the statistical data provided by EDF PRA data handbooks using traditional statistical tools such as Maximum Likelihood Estimation (MLE). Therefore, when new data become available, statistical tools are first used to provide estimated values from the pooled data (e.g. nominal values and confidence intervals) from which belief functions for uncertainty representation are constructed. However, this approach is only appropriate to the case where data come from the operating feedback observations and when the number of observations is sufficiently large. Furthermore, if additional sources of information are to be incorporated, such as expert's opinions, the pooling data of MLE has limits to account for these kinds of information. The expert's opinions are often used in the context of PRA model for the events whose the frequency of occurrence is very small i.e. rarely or never observed. Therefore, in order to incorporate the experts' opinions with the available operating feedback data, two alternative perspectives for constructing belief functions in a more effective way are studied in this paper. The updated belief functions are built by combining the belief functions given each data. In doing so, the incorporation of other sources of information, such as expert's opinions will be done in a natural manner. The two proposed approaches also allow us to deal with the prior ignorance in a more appropriate manner than the classical way. In the first approach, we still use the MLE but in a different way. For each independent serie of observations, the belief functions are firstly built from the confidence intervals provided by MLE, and then the updated belief functions are obtained by using the Dempster's rule of combination (ROC) to aggregate all the belief functions. In the same manner but within the perspective of Bayesian theorem, the second approach relies on the General Bayesian Theorem (GBT) to provide belief functions given each data. The GBT introduced by Smets in [13] performs the same task as the classical Bayesian theorem but within the context of belief functions instead of probability functions. This theorem and the pignistic transformation are the essential tools of the so-called Transferable Belief Model (TBM) which is a subjective interpretation of the DST [14]. The main objective of this paper is to study the use of these approaches for updating belief functions in the context of nuclear PRA data.

The section 2 of this article presents shortly basic notions of the Dempster–Shafer theory of belief functions. In section 3, the updating of belief functions with the ROC and the GBT is presented. The section 4 studies the application of these two approaches in the context of PRA. The comparison of these two approaches with the currently used method is carried out through a practical example in the section 5. In the section 6, some conclusions and perspective are finally given.

2 The Dempster-Shafer Theory of Belief Functions

The Dempster-Shafer Theory of evidence [6], also known as the theory of belief functions, is a generalization of the Bayesian theory of subjective probability in that it allows less restrictive assumptions about the likelihood than in the case of probabilistic characterization of uncertainty. In literature, this theory has been used in risk assessment for industrial applications [11][17][18] and recently studied in the context of PRA for treating the uncertainty [20][21]. In this framework, the epistemic uncertainty associated to the input parameters of PRA is no longer characterized by a single probability distribution but by so-called belief and plausibility functions. In doing so, we can avoid the problem of choosing an appropriate probability distribution for uncertainty representation in a context of lack of data. The definition of these functions is shortly outlined now.

Let $\Theta=\{\theta_1, \theta_2,\ldots,\theta_N\}$ be a finite set of possible values for parameter θ called the frame of discernment. Unlike the probability distribution which is completely defined by the weight of each singleton θ_i, the belief functions are defined on the set of subsets of Θ, called 2^Θ. In the DST, the basic measure is represented by a so-called basic belief assignment

$$m: 2^{\Theta} \rightarrow [0,1] \qquad \sum_{A\subseteq\Theta} m(A) = 1 \tag{1}$$

Where $m(\Theta)=1$ and $m(\emptyset)=0$. The basic belief assignment (BBA) $m(A)$ represents the degree of belief that the actual solution is exactly committed to A and due to lack of knowledge cannot be attributed any more specific event. The state of complete ignorance is represented by the so-called vacuous BBA defined by $m(\Theta)=1$ that is no information is available for the more likely values among Θ. A Bayesian BBA is a BBA whose focal sets are singletons. A BBA is said to be consonant if its focal sets are nested.

The belief function *Bel*, the plausibility function *Pl* and the commonality function *q* are defined for all $B \subseteq \Theta$ as follows

$$Bel(B) = \sum_{A\subseteq B} m(A) \tag{2}$$

$$Pl(B) = \sum_{A\cap B\neq\emptyset} m(A) \tag{3}$$

$$q(B) = \sum_{A\supseteq B} m(A) \tag{4}$$

The belief $Bel(B)$ obtained by the summation of BBAs for all elements A which are fully included in proposition B expresses the "total" degree of belief. The degree of plausibility $Pl(B)$ is calculated by adding BBAs of elements A whose the intersection with proposition B is not an empty set. The commonality function q is used for mathematical purposes only. In the perspective of Walley [16], these belief and plausibility functions consist of lower and upper bounding probability functions of the true but unknown probability distribution.

When a decision needs to be made, we use a so-called pignistic(1) transformation which induces a pignistic probability function from the belief functions. This is the

(1) *Pignistic* means '*bet*' in Latin

result of applying the TBM model introduced by Smets [14] which is a subjective interpretation of the DST. The TBM is a two-level mental model in which the beliefs are represented and quantified at the credal level by belief functions, whereas decision making is based on the probability distributions and takes place at the pignistic level. The use of the TBM model for decision making in the context of PRA has been studied in [21].

In the next sections, the Dempster-Shafer Theory is studied for the use of updating the belief functions when new evidence is available.

3 Approaches for updating belief within the Theory of Belief functions.

Combination of different sources of evidence is one of the important fields when dealing with uncertainty. The Dempster-Shafer Theory of belief functions offers many approaches for aggregating belief functions in a natural way. Two approaches often studied and used in some real applications are outlined hereafter. These two approaches allow the belief functions to be updated by taking account of the prior sources of information (e.g. experts' opinions or previous data) in addition with new available data.

In the following we consider a random variable X on the state space Ψ and characterized by its probability distribution P_θ, with the parameter θ taking its values in Θ.

3.1 Dempster's Rule of Combination (ROC)

Suppose that the uncertainty associated to the parameter of the model is characterized by belief functions. These functions need to be updated when new data on the space Ψ become available. If data observations are independently collected, the belief functions of the parameter given each data can be all combined together using the Dempster's rule of combination (ROC). Let BBA m_1 and BBA m_2 represent respectively the belief functions given the first data and the second data over the frame Θ, according to the ROC, then the combined BBA is calculated as follows

$$m_{12}(A) = (m_1 \oplus m_2)(A)$$
$$= \frac{1}{1-K} \sum_{A_1 \cap A_2 = A} m_1(A_1).m_2(A_2) \text{ for } \forall A \subseteq \Theta \quad (5)$$

Where $K = \sum_{A_1 \cap A_2 = \emptyset} m_1(A_1).m_2(A_2)$ is a measure of the amount of conflict between the two BBAs.

Therefore, by considering m_1 as the prior BBA and m_2 as the BBA given new available data, the posterior belief functions can be obtained using the above ROC. In some contexts, the prior information can be simply vacuous belief functions i.e. m(Θ)=1 which express the total ignorance.

As wee can see in equation (5), since the operator $\oplus$ used in this rule is both associative and commutative, thus the order of these functions to combine is not relevant. Note that when the belief functions are Bayesian functions, Shafer [7] proved that the Bayes' rule of conditioning is a special case of the Dempster's rule of combination.

3.2 Generalized Bayesian Theorem in TBM

The previous approach for aggregating the belief functions of the uncertain parameter θ involved a fairly standard application of DST. However, a generalization of the Bayes' rule within the TBM may be used to update the belief functions in a manner more closely aligned with updating of probability distributions via the classical Bayes' rule. This approach is now outlined.

3.2.1 Generalized Bayesian Theorem

As we know, in probability theory, the Bayesian theorem allows the computation of the posterior probability function of θ given observed realizations of X from the likelihood of X given θ and some prior probability distribution of θ. The same idea has been extended in the TBM context [13] where conditional belief functions of θ given observations of X is built from the conditional belief function of X given each $\theta_i \in \Theta$ and a vacuous prior belief of θ. Thus, if we know the conditional plausibilites $pl^\Psi(x|\theta_i)$ of X given each $\theta_i \in \Theta$ and according to the GBT, the conditional belief functions for all $A \subseteq \Theta$ given an observation $x \in \Psi$ are computed as follows:

$$m^\Theta(A|x) =$$
$$= C.\prod_{\theta_i \in A} pl^\Psi(x|\theta_i) \prod_{\theta_i \in \bar{A}} (1 - pl^\Psi(x|\theta_i)) \quad (6)$$

$$Bel^\Theta(A|x) =$$
$$= C\left(\prod_{\theta_i \in \bar{A}} (1 - pl^\Psi(x|\theta_i)) - \prod_{\theta_i \in \Theta} (1 - pl^\Psi(x|\theta_i))\right) \quad (7)$$

and

$$pl^\Theta(A|x) = C\left(1 - \prod_{\theta_i \in A} (1 - pl^\Psi(x|\theta_i))\right) \quad (8)$$

Where $C^{-1} = 1 - \prod_{\theta_i \in \Theta} (1 - pl^\Psi(x|\theta_i))$ is the normalized factor which is introduced when the assumption of closed-world is made i.e. the BBA $m(\emptyset) = 0$ is assumed. The interesting point in the GBT is that the needed prior belief on Θ is a vacuous belief function which is the perfect representation of total ignorance. We can thus avoid one of the delicate problems of classical Bayesian approach related to choosing an appropriate a priori. In the context of updating belief functions, the posterior beliefs can be obtained using the Dempster's rule of combination applied to the above conditional belief function given new data and the prior belief function built from the previous data.

In the case of having n independent series of observations with event counts x_1, x_2, … x_n resulting from the same probabilistic model (e.g. Poisson model), in order to aggregate belief functions given these observations, we can construct *n* conditional belief functions of θ given each event count x_i and then combine these belief functions by the ROC. The same result can be obtained in a different way by considering the joint conditional plausibility function $pl^{\Psi}(x_1,...,x_n \mid \theta_i)$ directly obtained from the joint observations (x_1,x_2,…,x_n) using the notion of "conditional cognitive independence" as proposed in [5][13]. As a result, the plausibility function $pl^{\Psi}(x_1,...,x_n \mid \theta_i)$ of observing the joint observation (x_1, x_2…x_n) given each $\theta_i \in \Theta$ is the product of the individual plausibility functions of all observations i.e.:

$$pl^{\Psi}(x_1,...x_n \mid \theta_i) = \prod_{k=1}^{n} pl^{\Psi}(x_k \mid \theta_i) \tag{9}$$

Then, the equations above (6,7,8) can be applied to calculate the conditional belief functions on Θ given the joint observation. This above property is essential and in fact the core of the axiomatic derivations of the GBT [12]. Let us now discuss about the performance of two ways for calculating the conditional belief functions given the data in GBT. From a computational point of view, the way of constructing conditional belief functions of θ given joint observations (x_1, x_2…x_n) is more efficient than calculating the conditional belief functions of θ given each x_i and combing them by ROC. This is because the former way is simply involved in the "product" operations (9) while the later concern with the orthogonal sums of ROC which require practically much more computational time. However, if we have some other sources of information such as expert's judgments or any source which is distinct from the observations resulted from the same random process of probabilistic model, the Dempster's rule would be more appropriate to use to construct the overall belief functions. This situation is often encountered in the context of PRA model.

As can be seen so far, the updating of the belief functions of the uncertain parameter θ of the probabilistic model $\{P_\theta : \theta \in \Theta\}$ using the GBT just requires to calculate the conditional plausibility functions $pl^{\Psi}(x \mid \theta_i)$ given each $\theta_i \in \Theta$. In the following paragraph we will discuss about the calculation of this conditional plausibility function.

3.2.2 About the calculation of the conditional plausibility functions $pl^{\Psi}(x \mid \theta_i)$

As we know, the probabilistic distribution of a random variable X describes the degree of chance (estimated by the long run frequency) of its independent realizations x_1, x_2,…,x_n. If the probability distribution of the random variable X is known then the Hacking's frequency principle [8] claims that the degree of belief of an event is equal to its probability i.e. $Bel=P_\theta$. However, in the TBM model, the degrees of chance are not equated with the degrees of belief. Thus, if asked about the belief held by an agent regarding the future realization of X, as argued in [1], this degree of belief should be distinguished from the degree of chance which is only handled at pignistic level in the TBM model. Hence, according to [1], "*we replace the Hacking's principle by the weaker requirement that pignistic probability of an event is considered as its long run frequency when the latter is known*". In other words, the belief functions on credal level quantifying the belief regarding the next realization of a random variable should be such that its pignistic probability distribution is the probabilistic model $\{P_\theta : \theta \in \Theta\}$. In order to be consistent with the underlying assumptions of the TBM used in our context, we will adopt in this paper this point of view to derive the beliefs with regard to the future observations of a random variable.

If the pignistic probability distribution equated with a probabilistic distribution is known while the corresponding belief and plausibility functions are unknown, then we can recover these functions using the least commitment principle proposed in [3]. Since the pignistic transformation is not bijective, an infinite number of BBA, called a set of isopignistic belief functions, can induce the same *BetP*. In the absence of additional information, the least commitment principle suggests to choose, in the set of all isopignistic BBA, the one that maximizes the commonality function *q*, named *q*-least committed (*q*-LC). Dubois, Prade and Smets [3] demonstrated that the (*q*-LC) BBA associated with a given pignistic probability distribution *BetP* is unique and consonant (i.e. a possibility distribution). Therefore, according to the results of [3], the conditional plausibility $pl^{\Psi}(x)$ of observing *x* over the discrete space Ψ given each $\theta_i \in \Theta$ is calculated from *BetP* as follows:

$$pl^{\Psi}(x) = \sum_{y \in \Psi} \min(p(x), p(y)) \tag{10}$$

Where $p(x)=BetP(x)$ which is a unimodal discrete probability distribution. In the case where Ψ is continuous, the conditional plausibility of a probability density is defined in the same way by substituting the finite sums by integrals.

After calculating the posterior belief functions, similarly to the classical way for updating a probability distribution with the Baye's theorem, it is possible to estimate the parameter by constructing the pignistic probability induced by the posterior belief functions.

In this section, we studied two approaches for updating the belief functions when new knowledge is available. The first approach is simply based on a standard application of the Dempster-Shafer theory while the second is based on the generalization of the Bayes' rule within the TBM. Both approaches do not require prior belief functions to be set. In literature, theses two approaches have been criticized by [16] and recently discussed in [4]. In practice, the use of Dempster's rule and GBT has been studied for updating the belief functions in some applications [5][11]. In the next sections, we will consider these two approaches in the context of nuclear PRA data.

4 Application of belief updating approaches to Nuclear PRA context

The use of belief functions for modeling the uncertainty associated to reliability parameters in the PRA context has been studied in [20][21]. In these works, the focal elements are constructed from the data as the closed intervals (focal intervals) and then the belief functions are derived. From a computational point of view, this construction is helpful to propagate the uncertainty through a given model function by simulation code. In this section, we will study the use of the approaches presented previously for updating belief functions when new data are available. But let us start by recalling the method currently used in this purpose and based on the MLE [20][21].

4.1 Belief updating from pooled data with Maximum Likelihood Estimation

The MLE is often used to estimate the value of parameters of probabilistic models given observations as the current practice of EDF's Nuclear PRA. Basically, this method relies on the principle of long run frequency to estimate the value of parameters given the number of observations over a time period. For example, the failure rate (often noted as λ) of a component with exponential lifetime is estimated by:

$$\hat{\lambda} = \frac{x}{t} \tag{11}$$

Where x is the number of observed failure events over the time period t. Associated with the estimator, the confidence interval is provided to represent the range of possible values of parameter in which the true value is contained "in most cases" (i.e. for a fraction 100(1-α) of the samples). In the practice of PRA, a 90% confidence interval is often used. When new observations become available, they are combined with previous ones using the pooled data technique to give an updated estimator and a new confidence interval. The new estimator is calculated as:

$$\hat{\lambda} = \frac{\sum_i x_i}{\sum_i t_i} \tag{12}$$

Where $\sum_i x_i$ is the total number of observations and $\sum_i t_i$ is the total exposure time. The confidence interval is also recalculated given this new information. In the traditional uncertainty analysis of PRA, on the basis of this information, a presumed probability distribution such as a log-normal distribution is used in the sense that the subjective probability will reflect our beliefs regarding the values of parameter. However, this point of view has been questioned due to the potential impact of the choice of probability distribution on the results of decision making. An approach using the belief functions of DST is proposed to overcome the issue as studied in [19]. The construction of these functions is based totally on the information given in the form of a nominal value (i.e. an estimated value) and a confidence interval. Obviously, the updating of belief functions when new information is available is not carried out by mean of an aggregation of degrees of belief. Such an approach may have difficulty to incorporate with other sources of information such as those given by expert's opinion. This problem can be addressed using the ROC presented in section 3. This approach allows integrating the prior information given by experts' opinions or past experiences in a natural way. We will see hereafter how this approach is used in the context of PRA data.

4.2 Belief updating with Dempster's rule of combination

When the information about the values of uncertain parameters comes from experts' judgments, the belief functions of DST are appropriate to represent the degrees of beliefs regarding the uncertainty. As independent expert's judgments are given, the combination of these sources of information can be done using the ROC. The same manner can be applied to the case where operating feedback data become available and new belief functions are calculated by taking account of this data as well as the information given by expert's judgments. In this case, the belief functions given the operating data are obtained from the MLE approach and then aggregated with those assessed from expert's judgments. Obviously, one may also apply the ROC for statistical independent data within the MLE context by constructing the belief functions obtained from the confidence intervals of MLE given each data and then aggregating all these functions to obtain the updated belief functions. However some precautions should be taken when using the ROC since the belief functions are constructed on the basis of the confidence intervals of MLE which are randomly derived from a random probabilistic process. This can lead to some cases where the BBA is equal to zero because these confidence intervals may not overlap each other, i.e. they are disjoint intervals each other. This problem can be only addressed if we admit that all the confidence intervals contain the true value of parameter although this is only true in "most of the cases" (e.g. 90% of chance). This is an unavoidable drawback of the approaches based on the intervals of confidence of MLE to construct belief functions. In [19] some other approaches for the combination of sources of evidence such as mixing or enveloping approaches can be applied for addressing this issue. However, these methods are not appropriate in this context because they tend to widen the uncertainty while we aim to construct the belief functions concentrated around the true value, as new information is available. The GBT inspired from the classical Bayes' rule could be more suitable to construct belief functions given statistical independent observations since this approach does not rely on the use of random confidence intervals of MLE.

4.3 Belief updating with Generalized Bayesian Theorem.

The classical Bayes' theorem has been studied for the parameter estimation and the updating of uncertainty probability distributions in the context of nuclear PRA as in [2][10]. The major issue of this approach resides in choosing an appropriate prior probabilistic distribution since results of an uncertainty analysis could be impacted by this choice. The GBT approach within the theory of belief functions presented in section 3.2 could be the solution to this problem and allows us moreover to characterize the epistemic uncertainty in a more appropriate manner. In general, the probabilistic models in PRA are often supposed to be known in order to characterize the random occurrence of accidents that may occur at nuclear power plants. Therefore, when the belief functions are used to represent epistemic uncertainty associated to its parameters, the updating of these belief functions using GBT can be carried out by considering these probabilistic models as pignistic probability distributions as discussed in section 3. The conditional plausibility $pl^{\Psi}(x|\theta_i)$ on the space of data Ψ given each $\theta_i \in \Theta$ is calculated using the least commitment principle. The probabilistic model in PRA can be divided into two principal types: discrete model and continuous models. However, since the information provided in PRA databook is often given in the form of number of observations, it is usually enough to consider the conditional plausibility $pl^{\Psi}(x|\theta_i)$ on the discrete space of data Ψ. Let us study for instance a Poisson model with an event rate λ[(2)] over an operational time t, the probability of having x accidental events over Ψ given the value of event rate $\lambda_i \in \Theta$ is given as follows:

$$p(x \mid \lambda_i) = e^{-\lambda_i t} \frac{(\lambda_i t)^x}{x!} \quad (13)$$

[(2)] *we use the notation λ instead of θ for this example to keep the same notation used in PRA practical example of the section 5.*

Therefore, when the evidence in form of x failures is available, the conditional plausibility $pl^{\Psi}(x \mid \lambda_i)$ is simply calculated by:

$$pl^{\Psi}(x \mid \lambda_i) = \sum_{y\in\Psi} \min(p(x \mid \lambda_i), p(y \mid \lambda_i)) \quad (14)$$

For example let us consider a frame of data $\Psi=\{x_1, x_2, x_3, x_4\}$, the Poisson model given a specified value of λ_i has the probability distribution such that $p(x_1)=0.3$, $p(x_2)=0.4$, $p(x_3)=0.2$ and $p(x_4)=0.1$. The conditional plausibility $pl^{\Psi}(x_3 \mid \lambda_i)$ of having x_3 failures according to equation (14) is

$$pl^{\Psi}(x_3|\lambda_i)=\min(0.2,0.3)+ \min(0.2,0.4)+ \min(0.2,0.2)+ + \min(0.2,0.1)=0.7.$$

Having calculated the conditional plausibility over space Ψ, the conditional belief functions for all subset $A \subseteq \Theta$ given any observation $x \in \Psi$ can be obtained using equations (7,8). Nevertheless, as we can see, these belief functions from theses equations are computed for the subsets of the discrete frame Θ while it is proposed to construct them on the basis of focal elements which are closed intervals (i.e. focal intervals) for uncertainty propagation in later [21]. Therefore, in order to allow us to update the belief functions using the GBT in our context, it is necessary to transform the belief functions defined on discrete frame to those defined on the real line. We propose for this purpose to build the "empirical" cumulative belief functions and then get the focal intervals from the discretization process. Some additional tasks need therefore to be performed. First of all, to get the discrete frame Θ of a continuous variable θ, we partition the frame Θ such that we have an increasing ordered set of $\theta_1, \theta_2,\ldots, \theta_N$. Then we apply the GBT approach using above equations (7,8) or (6) to calculate the conditional belief and plausibility functions for sets $\{\theta_1\}$, $\{\theta_1, \theta_2\}$, $\{\theta_1, \theta_2, \theta_3\}\ldots\{\theta_1, \theta_2, \theta_3,\ldots, \theta_N\}$. Since these are nested sets, we have always for the belief function (the same for the plausibility function) that $Bel(\{\theta_1\})\leq Bel(\{\theta_1,\theta_2\})\leq \ldots\leq Bel(\{\Theta\})=1$. Therefore, similarly to the discrete probability theory if we consider elements $\theta_1, \theta_2,\ldots, \theta_N$ as order statistics and previous belief (plausibility) values as cumulative probabilities then we can build the "empirical" cumulative belief functions on the frame Θ of the continuous variable θ by using a step function. Thus, let note B sets $\{\theta_1\}$, $\{\theta_1, \theta_2\}$, $\{\theta_1, \theta_2, \theta_3\}\ldots\{\theta_1, \theta_2, \theta_3,\ldots, \theta_N\}$, theses functions are expressed as follows

$$Bel^{\Theta}((-\infty,\theta]\|x) = \sum_{B\subseteq\Theta} Bel^{\Theta}(B\|x).1_{B\subseteq(-\infty,\theta]\subseteq\Theta} \quad (15)$$

and

$$Pl^{\Theta}((-\infty,\theta]\|x) = \sum_{B\subseteq\Theta} Pl^{\Theta}(B\|x).1_{B\cap(-\infty,\theta]\neq\emptyset} \quad (16)$$

where $1_A(x)$ equals one if x is in A and zero in the opposite.

These two functions could be considered as the bounds of a p-box because they are both non-decreasing functions from the real values into the interval [0,1] and the function $Bel^{\Theta}((-\infty,\theta]\,|\,x)$ is less than or equal to $Pl^{\Theta}((-\infty,\theta]\,|\,x)$ for every value of θ. By adopting this view, the Dempster-Shafer focal intervals can be approximately obtained using the discretization methods as described in [19]. The principle of discretization is illustrated in the Figure 1.

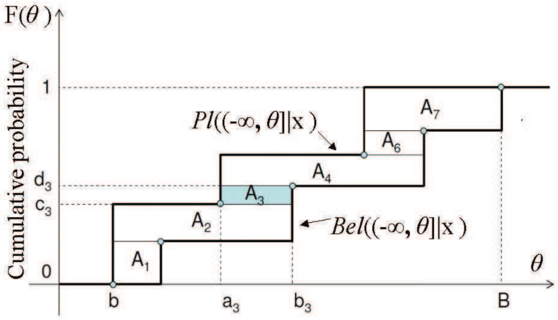

Figure 1 Principle of the construction of focal intervals from a p-box.

The lower and upper bounding functions are assumed to be right and left continuous, respectively. Each rectangle A_i in this figure corresponds to a focal interval $[a_i, b_i]$ with mass $([a_i,b_i])=d_i-c_i$ where d_i and c_i are probability values. These focal intervals then can be used to propagate the parameter uncertainty as done in the framework proposed in [21]. In summary, in order to use the GBT for updating the belief functions of an uncertain parameter θ of a PRA probabilistic model, we go through the following steps:

Step 1: Define the discrete frame Θ of possible values of uncertain parameter θ and then sort them in an increasing order for example, $\theta_1, \theta_2,\ldots,\theta_N$. In practice, the uncertain parameter θ is often given by a bounded confidence interval; the frame Θ can be obtained by discretizing this interval into N possible discrete values.

Step 2: When a new observation x_0 becomes available, compute the plausibilities $pl(x_0|\,\theta_i)$ of observing x_0 given each θ_i using the formula of *least commitment principle* (14).

Step 3: Use the *Generalized Bayesian Theorem*, to calculate the cumulative beliefs for θ_1, $\theta_2,\ldots,\theta_N$ and then construct "empirical" cumulative belief functions of equations (15,16) for each semi-closed interval $(-\infty, \theta_i]$ on the frame Θ given the observation x_0.

Step 4: Use the discretization methods to obtain focal intervals from "empirical" cumulative belief functions.

Step 5: If the belief functions of some other independent observations are available and/or the prior belief functions come from other sources (e.g. expert's judgment), the final posterior belief functions can be obtained using Dempster's rule of combination.

Step 6: When it is required to provide a point estimate value of parameter θ as in the PRA context, compute the mean (or median or mode) of the pignistic probability distribution induced from posterior belief functions.

In this section, we considered the application of updating belief functions for parameter uncertainty representation in the context of PRA. As we can see, since the mechanism of constructing the belief functions given new information of each method is quite different, thus the results obtained from each one could be different from one to another. Since our main goal is to build posterior belief and plausibility functions such that they should be concentrated around the true value of the parameter, the width between the belief function and the plausibility function should be reduced as new information are available. In order to measure this width of the belief functions obtained from each approach, the measure uncertainty as proposed in [12] can be applied. This measure is defined as follows

$$AW = \sum_{[a_i,b_i]_i\subseteq\Theta} m([a_i,b_i]).(b_i-a_i) \qquad (17)$$

This is called a non-specificity measure which quantifies the amount of uncertainty represented by belief functions. As we can see, it measures the aggregated width of all intervals which is the area between the belief and the plausibility functions. The smaller non-specificity measure *AW,* the more specific is the resulting of belief functions. In the following section, this measure will be employed to compare results of updating belief approaches through a practical example.

5 Practical example

In order to illustrate the above approaches through a practical example, we propose to take the example that has been used in [2]. The following example is addressed for the study of an initiating event of PRA but the principle can be applied for other types of failure events.

Problem: Considering a Poisson model with the true but unknown value of an initating event rate λ= 1.2 events per year (13.69E-5/h) over the time period of observation t= 6 years. Thus, the event count follows a Poisson distribution with mean λt = 7.2. In PRA, due to lack of data, the event rate λ is subjected to epistemic parameter uncertainty.

Suppose that we had already prior information about the event rate λ given by a point estimate and an error factor (EF[(3)]), say, λ_{mean}= 5E-5/h and EF=5 which can be

(3) *EF is often used in PRA context to indicate the range of possible values of an uncertain parameter.*

interpreted as the 90% confidence interval such as $\lambda \in [1E-5,\ 25E-5]$ see [20]. This prior information can be viewed as obtained from either expert's judgment or from previous experience. The prior belief functions based on this information are constructed by the approach studied in [2] by considering the point estimate λ_{mean} as the mean value of the uncertain variable λ. These belief functions are displayed in the Figure 2.

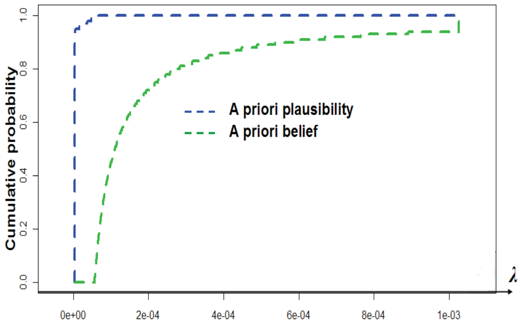

Figure 2 Prior belief and plausibility functions

Now let us suppose that we have new data that are observed from nuclear plants. This can be done by considering the above Poisson process be repeated in a number of times, say, 40 event counts are generated. These may be interpreted as counts from 40 identical plants, each observed for 6 years, or from 40 possible six-year periods at the same plant. Figure 3 shows that the first randomly generated event count was 10, the next was 5, the next was again 10, and so on. Some of the event counts were less than the long-term mean of 7.2, and some were greater. The maximum likelihood estimates of event rate λ are plotted as dots in Figure 3. The corresponding 90% confidence intervals for λ are also plotted. In the Figure 3, the vertical dashed line shows the true value of λ, 1.2.

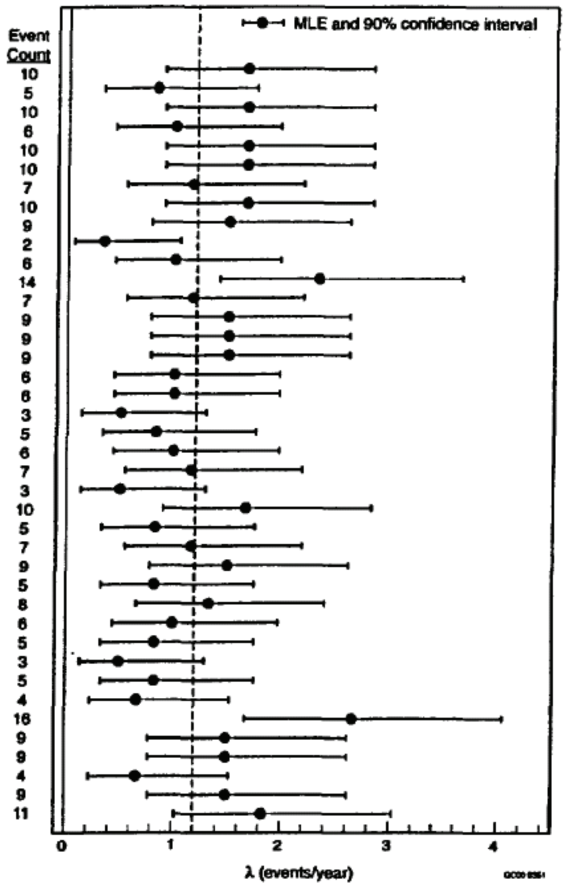

Figure 3 Confidence intervals from random data, all generated from the Poisson process [2].

Given new data, we will next construct the belief functions using the studied approaches. We will consider two cases: one observation and multiple independent observations.

Case 1: One observation

In this case, in order to show the advantage of the GBT with regard to the classical Bayes' theorem, we will distinguish the two following cases.

a. No prior information is available (prior ignorance)

Suppose that we have only the information about the initiating event from the first period of observation of the Poisson process which gives 10 event counts i.e. x=10. In this case, the point estimate value and the 90% confidence interval given by the MLE method are 19.02E-5/h (1.66/year) and [10.32E-5, 32.27E-5] respectively. The belief and plausibility functions can be constructed from this information as showed in the Figure 4.

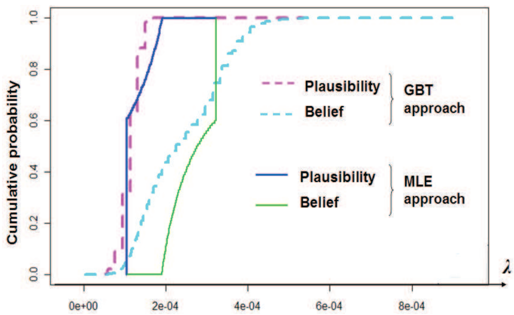

Figure 4 Belief and plausibility functions constructed from MLE approach and GBT without available prior information.

Since we have only one single data (i.e. the first period of observation) while no prior information is available, the Dempster's rule of combination of evidence is not necessary. The Figure 4 displays also the belief functions obtained from the GBT approach. Unlike the classical Baye's theorem where a prior probability distribution is required, no such requirement is needed in the GBT approach. In the absence of prior information, a vacuous belief function i.e. m(Θ)=1 which represents perfectly the total ignorance is sufficient. This allows us to avoid any assumption about the choice of an appropriate prior probability distribution as in the classical Bayes' theorem. As we can see from the Figure 4, compared to the belief functions of the MLE approach, the results of GBT in this case are more specific because the non-specificity measure AW (12.63E-5) is smaller that of the MLE (14.7E-5). The mean pignistic value of the GBT is 17.7E-5/h compared to true value of event rate (13.69E-5/h). Note that, the construction of belief functions is based on the information of 90% confidence interval which is viewed as the upper and lower bounds of the parameter. As discussed in [20][21], in some context this consideration may be helpful to eliminate the values outside the interval which are viewed as unrealistic. However, in other contexts, this can lead to loss of information. The results of the GBT approach are not impacted by this consideration.

b. A priori information is available

When prior information is available, the belief functions of this information can be combined with the belief functions given the new observations. Suppose that we have the prior information of *event rate* as from the Figure 2 i.e. λ_{mean}= 5E-5 and 90% confidence interval [1E-5, 25E-5]. This information is often given by experts' opinions, however, if desired, it can be also viewed as obtained from a previous observation. In this case, the point estimate λ_{mean}= 5E-5 can be considered as if the *event counts* over the time period of 6 year was 3. As a consequence, when the first data of the Poisson process comes with 10 event counts of the first period of observation, the event rate estimated from the pooled data by MLE approach is λ= (10+3)/(6+6)=1.083 events per year (12.36E-5/h) and the 90% confidence interval is [7.31E-5, 19.66E-5]. The belief functions constructed from the pooled data of MLE are showed in the Figure 5. On the other hand, instead of constructing the belief functions from the pooled data, we can use the ROC to build the belief functions given each data. In this case, it is merely sufficient to apply the ROC to prior belief functions (Figure 2) and the belief functions given the first new data (Figure 4). The same way applied to the conditional belief functions with GBT approach. The results of these approaches are displayed in the Figure 5.

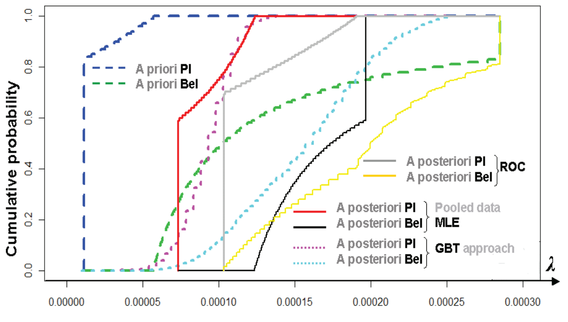

Figure 5 Posterior belief and plausibility functions of approaches vs. prior belief and plausibility functions.

As can be seen, the area between the belief and plausibility functions of GBT approach is smallest since its non-specificity measure (AW=6.37E-5) is smaller than that of pooled data MLE approach (8.33E-5) and that of ROC approach (8.35E-5). The mean pignistic value of GBT approach is 12.48E-5/h compared to this value given ROC approach (15.8E-5/h). These values are not far from the true value (13.69E-5/h).

In the first case study, we considered that we had only one data from the first period of observation. In the next case, we suppose having multiple independent observations.

Case 2: Multiple independent observations

In this case, suppose that we have 10 independent series of observations which are collected either from 10 identical power plants during the same time period or from 10 possible six-year periods at the same plant. Thus we have a series of event counts (10,5,10,6,10, 10,7,10,9,2). In the Figure 3, we use the first ten event counts among 40 event counts generated from the random Poisson process.

As in case 1b, given these observations in conjunction with the prior information, the pooled data MLE approach gives the estimated value 14.18E-5/h and 90% confidence intervals [11.70E-5, 17.04E-5]. The belief functions constructed from the pooled data are showed in the Figure 6. In this figure, the results of the ROC approach are also displayed. As can be seen, the resulting belief and plausibility functions do not contain the true

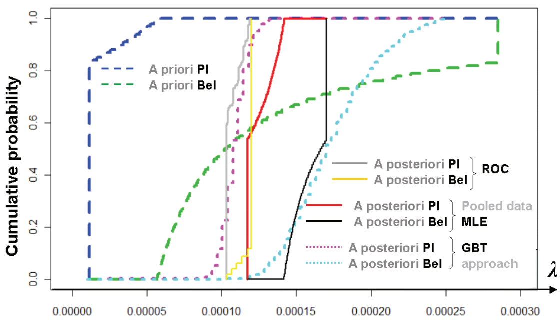

Figure 6 Posterior belief and plausibility functions of approaches in case of multiple independent observations.

value of the event rate because the highest value of these functions on horizontal axe (the maximum value) is smaller than 13.69E-5/h. This is explained by the fact that the assumption that all 90% confidence intervals must contain the true value of parameter is not verified (see the 10^{th} event count).The belief and plausibility functions coming from the GBT are slightly less specific than those coming from the MLE approach in this case but the results allow taking into account possible values located outside the 90% confidence interval of MLE.

6 Conclusions and perspective

In this paper, we studied different approaches for updating belief functions representing parameter uncertainty given new available information in the context of PRA. Although the method of constructing belief functions from pooled data of MLE is intuitive and consistent with the current practice of EDF PRA data, it has some drawbacks regarding the incorporation with other sources of information such experts onions. The method of using the ROC to aggregate belief functions given data within the MLE context is not recommended since its results are too sensitive to random sampling process. The GBT approach appears to be the most appropriate approach to use in PRA context. This approach can address the major issue in the classical Baye's rule about the assumption of prior probability distribution and moreover allows us to overcome the existing drawbacks associated to the MLE approach.

The use of DST for uncertainty representation has been only recently studied in PRA context. A number of challenges of this framework come up for its application within the industrial risk analysis. These approaches studied in this paper for constructing and updating the belief functions need to be reviewed in PRA community and studied within industrial contexts to be integrated in the formal regulatory process.

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