Non-conflicting and Conflicting Parts of Belief Functions

Milan Daniel
Institute of Computer Science, Academy of Sciences of the Czech Republic
milan.daniel@cs.cas.cz

Abstract

Non-conflicting and conflicting parts of belief functions are introduced in this study. The unique decomposition of a belief function defined on a two-element frame of discernment to non-conflicting and indecisive conflicting belief function is presented. Several basic statements about algebra of belief functions on a general finite frame of discernment are introduced and unique non-conflicting part of a BF on an n-element frame of discernment is presented here.

Keywords. belief function, Dempster-Shafer theory, Dempster's semigroup, conflict between belief functions, uncertainty, non-conflicting part of belief function, conflicting part of belief function.

1 Introduction

Belief functions are one of the widely used formalisms for uncertainty representation and processing that enables representation of incomplete and uncertain knowledge, belief updating, and combination of evidence. They were originally introduced as a principal notion of the Dempster-Shafer Theory or the Mathematical Theory of Evidence [17].

When combining belief functions (BFs) by the conjunctive rules of combination, conflicts often appear, which are assigned to $\emptyset$ by un-normalized conjunctive rule $\circledcirc$ or normalized by Dempster's rule of combination $\oplus$. Combination of conflicting BFs and interpretation of conflicts is often questionable in real applications, thus a series of alternative combination rules was suggested and a series of papers on conflicting belief functions was published, e.g. [2, 5, 16, 19].

In [9], new ideas concerning interpretation, definition, and measurement of conflicts of BFs were introduced. We presented three new approaches to interpretation and computation of conflicts: combinational conflict, plausibility conflict, and comparative conflict. Differences were made between conflicts between BFs and internal conflicts of single BF; a conflict between BFs was distinguished from the difference between BFs.

When analyzing mathematical properties of the three approaches to conflicts of BFs in [10], there appears a possibility of expression of a BF Bel as Dempster's sum of non-conflicting BF Bel_0 with the same plausibility decisional support as the original BF Bel has and of indecisive BF Bel_S which does not prefer any of the elements of frame of discernment. The presented contribution analyses existence and uniqueness of such BFs Bel_0 and Bel_S.

The study starts with belief functions and algebraic preliminaries in Section 2. The situation on 2-element frame (Section 3) is followed by a study of a/the case of general finite frames of discernment (Section 4). Some comments on alternative rules of belief combination are presented in Section 5.

2 Preliminaries

2.1 General Primer on Belief Functions

We assume classic definitions of basic notions from theory of *belief functions* (BFs) [17] on finite frames of discernment $\Omega_n = \{\omega_1, \omega_2, ..., \omega_n\}$, see also [4–9]; for illustration or simplicity, we often use 2- or 3-element frames Ω_2 and Ω_3. A *basic belief assignment (bba)* is a mapping $m : \mathcal{P}(\Omega) \longrightarrow [0,1]$ such that $\sum_{A \subseteq \Omega} m(A) = 1$; the values of the bba are called *basic belief masses (bbm)*. $m(\emptyset) = 0$ is usually assumed, then we speak about *normalized bba*. A *belief function (BF)* is a mapping $Bel : \mathcal{P}(\Omega) \longrightarrow [0,1]$, $Bel(A) = \sum_{\emptyset \neq X \subseteq A} m(X)$. A *plausibility function* $Pl(A) = \sum_{\emptyset \neq A \cap X} m(X)$. There is a unique correspondence among m and corresponding Bel and Pl thus we often speak about m as about belief function.

A *focal element* is a subset X of the frame of discernment, such that $m(X) > 0$. If all the focal elements are *singletons* (i.e. one-element subsets of Ω), then we speak about a *Bayesian belief function* (BBF), it is a probability distribution on Ω in fact. If all the focal elements are either singletons or whole Ω (i.e. $|X| = 1$ or $|X| = |\Omega|$), then we speak about a *quasi-Bayesian belief function* (qBBF), it is something like 'un-normalized probability distribution'. If all focal elements are nested, we speak about *consonant belief function.*

Dempster's (conjunctive) rule of combination $\oplus$ is given as $(m_1 \oplus m_2)(A) = \sum_{X \cap Y = A} K m_1(X) m_2(Y)$ for $A \neq \emptyset$, where $K = \frac{1}{1-\kappa}$, $\kappa = \sum_{X \cap Y = \emptyset} m_1(X) m_2(Y)$, and $(m_1 \oplus m_2)(\emptyset) = 0$, see [17]; putting $K = 1$ and $(m_1 \oplus m_2)(\emptyset) = \kappa$ we obtain the *un-normalized conjunctive rule of combination* $\textcircled{\scriptsize $\cap$}$, see e. g. [18]. *The disjunctive rule of combination* is given by the formula $(m_1 \textcircled{\scriptsize $\cup$} m_2)(A) = \sum_{X \cup Y = A} m_1(X) m_2(Y)$, see [12].

Yager's rule of combination $\textcircled{\scriptsize Y}$, see [21], is given as $(m_1 \textcircled{\scriptsize Y} m_2)(\emptyset) = 0$, $(m_1 \textcircled{\scriptsize Y} m_2)(A) = \sum_{X,Y \subseteq \Theta,\, X \cap Y = A} m_1(X) m_2(Y)$ for $\emptyset \neq A \subset \Theta$, and $(m_1 \textcircled{\scriptsize Y} m_2)(\Theta) = m_1(\Theta) m_2(\Theta) + \sum_{X,Y \subseteq \Theta,\, X \cap Y = \emptyset} m_1(X) m_2(Y)$;

Dubois-Prade's rule of combination $\textcircled{\scriptsize D}$ is given as $(m_1 \textcircled{\scriptsize D} m_2)(A) = \sum_{X,Y \subseteq \Theta,\, X \cap Y = A} m_1(X) m_2(Y) + \sum_{X,Y \subseteq \Theta,\, X \cap Y = \emptyset, X \cup Y = A} m_1(X) m_2(Y)$ for $\emptyset \neq A \subseteq \Theta$, and $(m_1 \textcircled{\scriptsize D} m_2)(\emptyset) = 0$, see [11].

We say that BF Bel is *non-conflicting* when conjunctive combination of Bel with itself does not produce any conflicting belief masses (when $(Bel \textcircled{\scriptsize $\cap$} Bel)(\emptyset) = 0$, i.e., $Bel \textcircled{\scriptsize $\cap$} Bel = Bel \oplus Bel$), i.e. whenever $Pl(\omega_i) = 1$ for some $\omega \in \Omega_n$. Otherwise, BF is *conflicting*, i.e., it contains some internal conflict [9].

Let us recall U_n the *uniform Bayesian belief function*[1] [9], i.e., the uniform probability distribution on Ω_n, and *normalized plausibility of singletons*[2] of Bel: the BBF (probability distribution) $Pl\_P(Bel)$ such, that $(Pl\_P(Bel))(\omega_i) = \frac{Pl(\{\omega_i\})}{\sum_{\omega \in \Omega} Pl(\{\omega\})}$ [3, 7].

Let us define an *indecisive (indifferent) BF* as a BF, which does not prefer any $\omega_i \in \Omega_n$, i.e., BF which gives no decisional support for any ω_i, i.e., BF such that $h(Bel) = Bel \oplus U_n = U_n$, i.e., $Pl(\{\omega_i\}) = const.$, i.e., $(Pl\_P(Bel))(\{\omega_i\}) = \frac{1}{n}$.

2.2 Belief Functions on 2-Element Frame of Discernment; Dempster's Semigroup

Let us suppose, that the reader is slightly familiar with basic algebraic notions like *a semigroup* (an algebraic structure with an associative binary operation), *a group* (a structure with an associative binary operation, with a unary operation of inverse, and with a neutral element), *a neutral element* n ($n * x = x$), *an absorbing element* a ($a * x = a$), *a homomorphism* f ($f(x * y) = f(x) * f(y)$), etc. (Otherwise, see e.g., [4, 6, 14, 15].)

We assume $\Omega_2 = \{\omega_1, \omega_2\}$, in this subsection. There are only three possible focal elements $\{\omega_1\}, \{\omega_2\}, \{\omega_1, \omega_2\}$ and any normalized *basic belief assignment (bba)* m is defined by a pair $(a, b) = (m(\{\omega_1\}), m(\{\omega_2\}))$ as $m(\{\omega_1, \omega_2\}) = 1 - a - b$; this is called *Dempster's pair* or simply *d-pair* in [4, 6, 14, 15] (it is a pair of reals such that $0 \leq a, b \leq 1, a + b \leq 1$).

Extremal d-pairs are the pairs corresponding to BFs for which either $m(\{\omega_1\}) = 1$ or $m(\{\omega_2\}) = 1$, i.e., $(1, 0)$ and $(0, 1)$. The set of all non-extremal d-pairs is denoted as D_0; the set of all non-extremal *Bayesian d-pairs* (i.e. d-pairs corresponding to Bayesian BFs, where $a + b = 1$) is denoted as G; the set of d-pairs such that $a = b$ is denoted as S (set of indecisive[3] d-pairs), the set where $b = 0$ as S_1, and analogically, the set where $a = 0$ as S_2 (simple support BFs). Vacuous BF is denoted as $0 = (0, 0)$ and there is a special BF (d-pair) $0' = (\frac{1}{2}, \frac{1}{2})$, see Figure 1.

The *(conjunctive) Dempster's semigroup* $\mathbf{D}_0 = (D_0, \oplus, 0, 0')$ is the set D_0 endowed with the binary operation $\oplus$ (i.e. with the Dempster's rule) and two distinguished elements 0 and $0'$. Dempster's rule can be expressed by the formula $(a, b) \oplus (c, d) = (1 - \frac{(1-a)(1-c)}{1-(ad+bc)}, 1 - \frac{(1-b)(1-d)}{1-(ad+bc)})$ for d-pairs [14]. In D_0 it is defined further: $-(a, b) = (b, a)$, $h(a, b) = (a, b) \oplus 0' = (\frac{1-b}{2-a-b}, \frac{1-a}{2-a-b})$, $h_1(a, b) = \frac{1-b}{2-a-b}$, $f(a, b) = (a, b) \oplus (b, a) = (\frac{a+b-a^2-b^2-ab}{1-a^2-b^2}, \frac{a+b-a^2-b^2-ab}{1-a^2-b^2})$; $(a, b) \leq (c, d)$ iff $[h_1(a, b) < h_1(c, d)$ or $h_1(a, b) = h_1(c, d)$ and $a \leq c]$ [4].

The principal properties of $\mathbf{D}_0$ are summarized by the following theorem:

Theorem 1 *(i) The Dempster's semigroup* $\mathbf{D}_0$ *with the relation* $\leq$ *is an ordered commutative (Abelian) semigroup with the neutral element* 0*;* $0'$ *is the only non-zero idempotent of* $\mathbf{D}_0$*.*
(ii) $\mathbf{G} = (G, \oplus, -, 0', \leq)$ *is an ordered Abelian group, isomorphic to the group of reals with the usual ordering. Let us denote its negative and positive cones as* $G^{\leq 0'}$ *and* $G^{\geq 0'}$*.*
(iii) The sets S, S_1, S_2 *with the operation* $\oplus$ *and the ordering* $\leq$ *form ordered commutative semigroups with neutral element* 0*; they are all isomorphic to the*

[1] U_n which is idempotent w.r.t. Dempster's rule $\oplus$, and moreover neutral on the set of all BBFs, is denoted as ${}_{nD}0'$ in [7], $0'$ comes from studies by Hájek & Valdés.

[2] Plausibility of singletons is called *contour function* by Shafer in [17], thus $Pl\_P(Bel)$ is a normalization of contour function in fact.

[3] BFs (a, a) from S are called *indifferent* BFs by Haenni [13].

[4] Note, that $h(a, b)$ is an abbreviation for $h((a, b))$, similarly for $h_1(a, b)$ and $f(a, b)$.

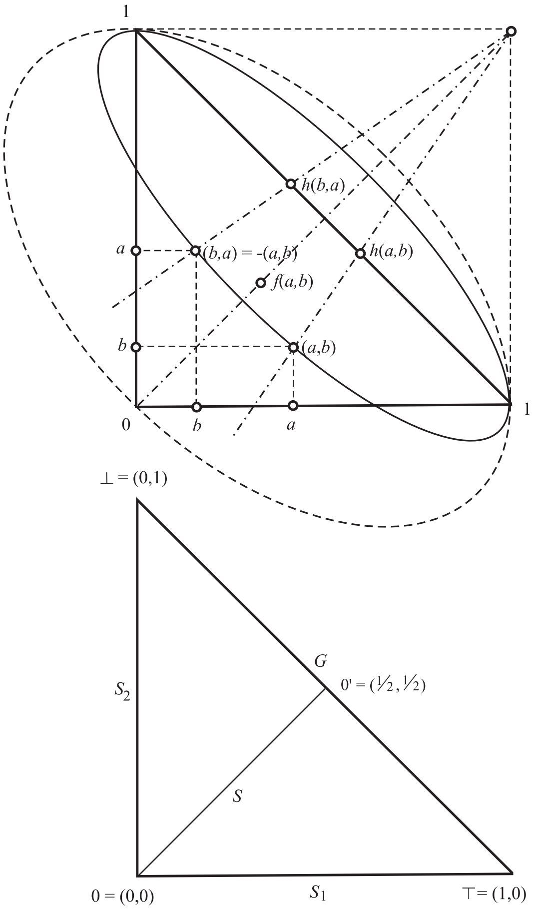

Figure 1: Dempster's semigroup D_0. Homomorphism h is in this representation a projection to group G along the straight lines running through the point $(1,1)$. All the Dempster's pairs lying on the same ellipse are mapped by homomorphism f to the same d-pair in semigroup S.

positive cone of the group of reals.
(iv) h is ordered homomorphism: $(D_0, \oplus, -, 0, 0', \leq) \longrightarrow (G, \oplus, -, 0', \leq)$*;* $h(Bel) = Bel \oplus 0' = Pl\_P(Bel)$*, i.e., the normalized plausibility probabilistic transformation.*
(v) f is homomorphism: $(D_0, \oplus, -, 0, 0') \longrightarrow (S, \oplus, -, 0)$*; (but, not an ordered one).*

For proofs see [14, 15, 20]. Let us denote $h^{-1}(a) = \{x \,|\, h(x) = a\}$ and similarly $f^{-1}(a) = \{x \,|\, f(x) = a\}$. Using the theorem, see (iv) and (v), we can express $\oplus$ as:

$$(a \oplus b) = h^{-1}(h(a) \oplus h(b)) \cap f^{-1}(f(a) \oplus f(b)).$$

Let us denote $D_0^{\geq 0} = \{(a,b) \in D_0 \,|\, (a,b) \geq 0\}$ and analogically $D_0^{\leq 0'} = \{(a,b) \leq 0'\}$.

2.3 BFs on n-Element Frames of Discernment

Analogically to the case of Ω_2, we can represent a BF on any n-element frame of discernment Ω_n by an enumeration of its m values (bbms), i.e., by a $(2^n - 2)$-tuple $(a_1, a_2, ..., a_{2^n-2})$, or as a $(2^n - 1)$-tuple $(a_1, a_2, ..., a_{2^n-2}; a_{2^n-1})$ when we want to explicitly mention also the redundant value $m(\Omega) = a_{2^n-1} = 1 - \sum_{i=1}^{2^n-2} a_i$. For BFs on Ω_3 we use $(a_1, a_2, ..., a_6; a_7) = (m(\{\omega_1\}), m(\{\omega_2\}), m(\{\omega_3\}), m(\{\omega_1, \omega_2\}), m(\{\omega_1, \omega_3\}), m(\{\omega_2, \omega_3\}); m(\{\Omega_3\}))$.

Unfortunately, no algebraic analysis of BFs on Ω_n for $n > 2$ has been presented till now.

3 Non-conflicting and Conflicting Parts of Belief Functions on 2-Element Frames of Discernment

For BFs on Ω_2 the following holds true:

Proposition 1 *BF Bel on* Ω_2 *is non-conflicting iff* $Bel \in S_1 \cup S_2$*.*

Proof. Obviously the simple support elements of S_1, S_2 are non-conflicting. $Pl(\{\omega_i\}) = m(\{\omega_i\}) + m(\{\omega_1, \omega_2\}) = 1 - m(\{\omega_j\})$, where $i \neq j$. Thus $Pl(\{\omega_i\}) = 1$ iff $m(\{\omega_j\}) = 0$ iff $Bel \in S_1 \cup S_2$. □

We will use the important property of Dempster's sum, which is respecting the homomorphisms h and f, i.e., respecting the h-lines and f-ellipses, when two BFs are combined on two-element frame of discernment [4, 14, 15]. Using this property we obtain the following statement.

Proposition 2 *Any belief function* $(a,b) \in \Omega_2$ *is the result of Dempster's combination of BF* $(a_0,b_0) \in S_1 \cup S_2$ *and a BF* $(s,s) \in S$*, such that* (a_0,b_0) *has the same plausibility decision support (same normalized plausibility) for the elements of* Ω_2 *as* (a,b) *does. (Trivially,* $(s,s) = (0,0) \oplus (s,s)$ *for* $(s,s) \in S$*, and* $(a_0,b_0) = (a_0,b_0) \oplus (0,0)$ *for elements of* S_1 *and* S_2*).*

$(a_0,b_0) \in S_1 \cup S_2$ *has no internal conflict, and* (s,s) *does not prefer any of the elements of* Ω_2*. Let us call* (a_0,b_0) *a* non-conflicting part *of* (a,b)*. There is* $(a_0,b_0) = (\frac{a-b}{1-b}, 0)$ *for* $a \geq b$ *and* $(a_0,b_0) = (0, \frac{b-a}{1-a})$ *for* $a \leq b$*.*

Proof. (a_0,b_0) is the intersection of h-line containing (a,b) with $S_1 \cup S_2$. Semigroup S is a part of h-line containing 0 and $0'$, thus the result of combination of any element $(s,s) \in S$ with (a_0,b_0), i.e, $(s,s) \oplus (a_0,b_0)$ lies on the same h-line as both (a_0,b_0) and (a,b).

$Pl\_P(a,b) = Pl\_P(a_0,b_0)$, thus $\frac{1-b}{2-a-b} = \frac{1-b_0}{2-a_0-b_0}$ and $\frac{1-a}{2-a-b} = \frac{1-a_0}{2-a_0-b_0}$. For $a \geq b$ there is $b_0 = 0$ and $\frac{1-b}{2-a-b} = \frac{1}{2-a_0}$, thus $\frac{2-a-b}{1-b} = \frac{2-a_0}{1}$, and $a_0 = 2 - \frac{2-a-b}{1-b} = \frac{a-b}{1-b}$. And similarly for $a \leq b$ there is $a_0 = 0$ and $\frac{1-a}{2-a-b} = \frac{1}{2-b_0}$, thus $b_0 = \frac{b-a}{1-a}$. □

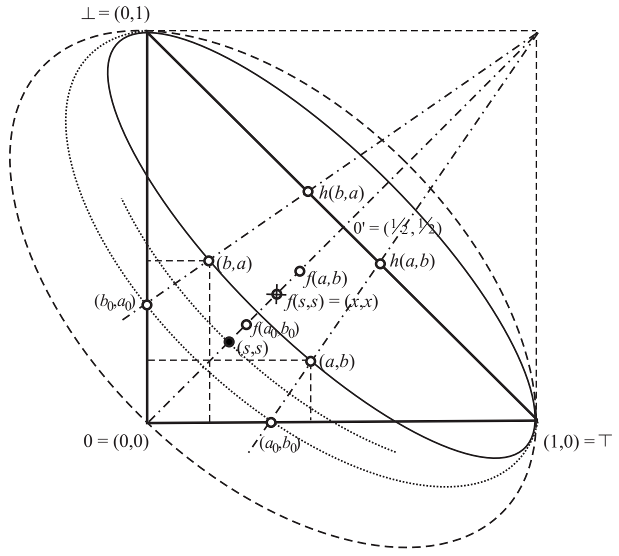

Figure 2: Conflicting and non-conflicting parts of BF on 2-element frame of discernment.

Let us look for (s,s) from the proposition now. It holds true that $(a,b) = (a_0,b_0) \oplus (s,s)$, thus it also holds true $f(a,b) = f(a_0,b_0) \oplus f(s,s)$. Let us denote $f(a_0,b_0) = (u,u)$, $f(a,b) = (v,v)$, $f(s,s) = (x,x)$ for a moment, thus we have $(u,u) \oplus (x,x) = (v,v)$, where $v = 1 - \frac{(1-u)(1-x)}{1-2ux} = \frac{u+x-3ux}{1-2ux}$, hence $u + x - 3ux = v - 2vux$ and $x = \frac{v-u}{1-3u+2uv}$. We can express this as Lemma 1 (i).

The existence of (x,x), thus also a possibility of its computation from (v,v) and (u,u) follows the fact, that S is isomorphic to the positive cone of group of reals, or a property subtraction in S as a substructure of algebraic structure dempsteroid [14, 15].

We already can compute value $f(s,s)$, the rest is computation of (s,s) as S-preimage of $f(s,s) = (s,s) \oplus (s,s) = (x,x)$. Similarly as before we have $x = 1 - \frac{(1-s)(1-s)}{1-2ss} = \frac{2s-3s^2}{1-2s^2}$ now, thus $2s - 3s^2 = x - 2s^2x$ and $0 = (3-2x)s^2 - 2s + x = 0$, hence $s_{1,2} = \frac{2 \pm \sqrt{4-4(3-2x)x}}{2(3-2x)} = \frac{1 \pm \sqrt{(1-x)(1-2x)}}{3-2x}$.

We know that $0 \leq s \leq x \leq \frac{1}{2}$, thus $0 \leq \sqrt{(1-x)(1-2x)} \leq 1$, $0 \leq 1 \pm \sqrt{(1-x)(1-2x)}$, $2 \leq 3 - 3x$. Thus $0 \leq \frac{1 \pm \sqrt{(1-x)(1-2x)}}{3-2x}$ always holds true. It should further hold true that $\frac{1 \pm \sqrt{(1-x)(1-2x)}}{3-2x} \leq \frac{1}{2}$, thus $2 \pm 2\sqrt{(1-x)(1-2x)} \leq 3 - 2x$ and $\pm 2\sqrt{(1-x)(1-2x)} \leq 1 - 2x$. It always hods true that $-\sqrt{(1-x)(1-2x)} \leq 0 \leq 1 - 2x$ for $0 \leq x \leq \frac{1}{2}$. On the other hand, from $2\sqrt{(1-x)(1-2x)} \leq 1 - 2x$, $4(1-x)(1-2x) \leq (1-2x)(1-2x)$, $4(1-x) \leq (1-2x)$, $3 \leq (2x)$ and $\frac{3}{2} \leq x$; this is in contradiction with $x \leq \frac{1}{2}$, hence it must be $s = \frac{1-\sqrt{(1-x)(1-2x)}}{3-2x}$.

We can formulate this as Lemma 1(ii). Finally, we obtain a summarization in Theorem 2.

Lemma 1 *(i) For any BFs (u,u), (v,v) on S, such that $u \leq v$, we can compute their Dempster's 'difference' (x,x) such that $(u,u) \oplus (x,x) = (v,v)$, as $(x,x) = (\frac{v-u}{1-3u+2uv}, \frac{v-u}{1-3u+2uv})$.*

(ii) For any BF (w,w) on S, we can compute its Dempster's 'half' (s,s) such that $(s,s) \oplus (s,s) = (w,w)$, as $(s,s) = (\frac{1-\sqrt{1-3w+2w^2}}{3-2w}, \frac{1-\sqrt{1-3w+2w^2}}{3-2w}) = (\frac{1-\sqrt{(1-w)(1-2w)}}{3-2w}, \frac{1-\sqrt{(1-w)(1-2w)}}{3-2w})$.

(iii) There is no Dempster's 'difference' on D_0 in general.

Proof. Parts (i) and (ii) were already proved by deriving of formulas for computing of (x,x) and (s,s). Nevertheless, we can alternatively verify the formulas is it follows.

$(a,b) \oplus (c,d) = (1 - \frac{(1-a)(1-c)}{1-(ad+bc)}, 1 - \frac{(1-b)(1-d)}{1-(ad+bc)})$ in general, for $a = b$ and $c = d$ we obtain a special case of the formula: $(a,a) \oplus (c,c) = (1 - \frac{(1-a)(1-c)}{1-(2ac)}, 1 - \frac{(1-a)(1-c)}{1-(2ac)})$.

$(u,u) \oplus (\frac{v-u}{1-3u+2uv}, \frac{v-u}{1-3u+2uv}) = (1 - \frac{(1-u)(1-\frac{v-u}{1-3u+2uv})}{1-(2u\frac{v-u}{1-3u+2uv})}, 1 - \frac{\frac{(1-u)(v-2u+2uv-v)}{1-3u+2uv}}{\frac{1-3u+2uv-2u(v-u)}{1-3u+2uv}}) = (\frac{-3uv+v+2u^2v}{1-3u+2u^2}, \frac{v(1-3u+2u^2)}{1-3u+2u^2}) = (v,v)$.

$(s,s) \oplus (s,s) = (1 - \frac{(1-s)^2}{1-(2s^2)}, \frac{2s-3s^2}{1-(2s^2)}) = (\frac{2\frac{1-\sqrt{(1-w)(1-2w)}}{3-2w} - 3(\frac{1-\sqrt{(1-w)(1-2w)}}{3-2w})^2}{1-(2(\frac{1-\sqrt{(1-w)(1-2w)}}{3-2w})^2)}, \frac{2\frac{1-\sqrt{}}{3-2w} - 3\frac{1-2\sqrt{}+(1-w)(1-2w)}{(3-2w)^2}}{1-(2\frac{1-2\sqrt{}+(1-w)(1-2w)}{(3-2w)^2})}) = (\frac{5w+4w\sqrt{}-6w^2}{5-6w+4\sqrt{}}, \frac{w(5+4\sqrt{}-6w)}{5-6w+4\sqrt{}}) = (w,w)$.

(iii) There is a lot of counter-examples, e.g., BFs Bel_1 and Bel_2 on the same f-ellipse: when combining any BF different from $0 = (0,0)$ with any of them, the result is on a narrower ellipse closer to G. □

Theorem 2 *Any BF (a,b) on 2-element frame of discernment Ω_2 is Dempster's sum of its unique* non-conflicting part *$(a_0,b_0) \in S_1 \cup S_2$ and of its unique* conflicting part *$(s,s) \in S$, which does not prefer any element of Ω_2, i.e. $(a,b) = (a_0,b_0) \oplus (s,s)$. It holds true that $s = \frac{b(1-a)}{1-2a+b-ab+a^2} = \frac{b(1-b)}{1-a+ab-b^2}$ and $(a,b) = (\frac{a-b}{1-b}, 0) \oplus (s,s)$ for $a \geq b$; and similarly that $s = \frac{a(1-b)}{1+a-2b-ab+b^2} = \frac{a(1-a)}{1-b+ab-a^2}$ and $(a,b) = (0, \frac{b-a}{1-a}) \oplus (s,s)$ for $a \leq b$.*

Proof. The existential part of the statement simply follows proposition 2 and both parts of Lemma 1. Uniqueness follows proposition 1, uniqueness of the h-line containing (a,b) and of its intersection with $S_1 \cup S_2$, and uniqueness of f-ellipse containing (a,b) and of its intersection with S. The rest is direct computation or verification. A verification for $a \geq b$ follows:
$(a_0, b_0) \oplus (s,s) = (\frac{a-b}{1-b}, 0) \oplus (\frac{b(1-b)}{1-a+ab-b^2}, \frac{b(1-b)}{1-a+ab-b^2}) = (1 - \frac{(1-\frac{a-b}{1-b})(1-\frac{b(1-b)}{1-a+ab-b^2})}{1-\frac{a-b}{1-b}\cdot\frac{b(1-b)}{1-a+ab-b^2}}, 1 - \frac{(1-\frac{b(1-b)}{1-a+ab-b^2})}{\frac{(1-b)(1-a+ab-b^2)}{(1-b)(1-a+ab-b^2)} - \frac{(a-b)b(1-b)}{(1-b)(1-a+ab-b^2)}}) = (\frac{a(1-b)}{(1-b)}, \frac{b(1-a)}{(1-a)}) = (a,b)$.
For $a \leq b$ we have:
$(a_0, b_0) \oplus (s,s) = (0, \frac{b-a}{1-a}) \oplus (\frac{a(1-a)}{1-b+ab-a^2}, \frac{a(1-a)}{1-b+ab-a^2}) = \ldots$, a and b and components of the couple are mutually substituted w.r.t. the case $a \geq b$, thus the result is (a,b) again. For equality of both formulas for s see [10]. □

An alternative proof is a derivation of formulas which is based on a similar idea as the derivation of formulas in Lemma 1. As we know the existence of (s,s) and that $a_0 = \frac{a-b}{1-b}$ for $a \geq b$, we know that $(a,b) = (a_0, 0) \oplus (s,s) = (1 - \frac{(1-a_0)(1-s)}{1-(a_0 s+0)}, 1 - \frac{(1-0)(1-s)}{1-a_0 s})$. Thus $a = 1 - \frac{(1-a_0)(1-s)}{1-(a_0 s+0)} = \frac{s+a-b-2as+bs}{1-b-as+bs}$. Hence $a - ab - a^2 s + abs = s + a - b - 2as + bs$ and $s = \frac{b(1-a)}{1-2a+b-ab+a^2}$. Similarly we have $b = 1 - \frac{(1-0)(1-s)}{1-a_0 s} = \frac{s-as}{1-b-as+bs}$. Hence $s = \frac{b(1-b)}{1-a+ab-b^2}$. Analogically, we can compute both versions of s for the case where $a \leq b$, see [10]. □

We can summarize formulas from the theorem as $(a,b) = (a_0, b_0) \oplus (s,s) = (max(\frac{a-b}{1-b}, 0), max(\frac{b-a}{1-a}, 0)) \oplus (\frac{min(a,b)(1-min(a,b))}{1+ab-max(a,b)-min^2(a,b)}, \frac{min(a,b)(1-min(a,b))}{1+ab-max(a,b)-min^2(a,b)})$. And analogically for the second expression of s [10].

Proof. Just a verification for $a \geq b$, and that for $a \leq b$. □

4 Non-conflicting Part of BFs on General Finite Frames of Discernment

Let us turn our attention to a question of non-conflicting and conflicting parts of BFs defined on an n-element frame of discernment $\Omega_n = \{\omega_1, ..., \omega_n\}$. We start with a characterization of the set of non-conflicting BFs.

Proposition 3 *The set of non-conflicting BFs is just the set of all BFs such, that all focal elements of a BF have non-empty intersection.*

Consonant BFs are a special case of non-conflicting BFs.

Proof. $Pl(\{\omega_i\}) = 1$ for some $\omega_i \in \Omega$ iff $\omega_i \in X$ for all X such that $m(X) > 0$ iff $\omega_i \in \bigcup_{m(X)>0} X$ iff $\bigcup_{m(X)>0} X \neq \emptyset$.
The least focal element of a consonant BF is intersection of its focal elements; there are many non-conflicting BFs which are not consonant on Ω_n, $n > 2$, e.g., $(0,0,0,0.7,0.3,0;0)$ on Ω_3, i.e., $m(\{\omega_1, \omega_2\}) = 0.7, m(\{\omega_1, \omega_3\}) = 0.3$. □

We would like to verify that Theorem 2 holds true also for BFs defined on general finite frames, i.e., to verify the following hypothesis:

Hypothesis 1 *We can represent any BF Bel on n-element frame of discernment $\Omega_n = \{\omega_1, ..., \omega_n\}$ as Dempster's sum $Bel = Bel_0 \oplus Bel_S$ of non-conflicting BF Bel_0 and of indecisive conflicting BF Bel_S which has no decisional support, i.e. which does not prefer any element of Ω_n to the others, see Figure 3.*

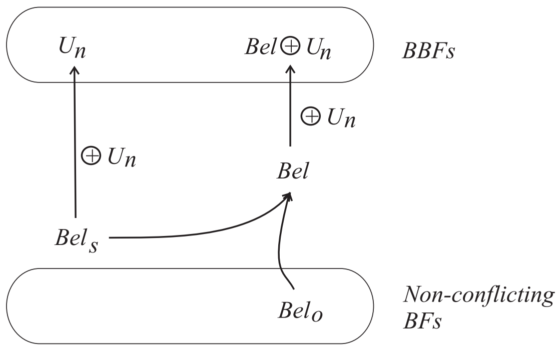

Figure 3: Schema of Hypothesis 1.

Similarly to two-element frames, we have simple trivial examples $Bel_N = Bel_N \oplus 0$ for all non-conflicting BFs Bel_N and $Bel_I = 0 \oplus Bel_I$ for all indecisive BFs Bel_I, where $0 = (0, 0, ..., 0; 1)$.

We would like to follow the idea from the case of two-element frames, see Figure 4. Unfortunately, there was not presented any algebraic description of BFs defined on n-element frames till now. We have nothing like Dempster's semigroup for n-element frames, we have no n-versions of $-Bel$ and of homomorphisms f and h, neither group properties of a set of indecisive BFs.

An issue of homomorphism h is quite promising: $h(Bel) = Bel \oplus U_n = Pl\_P(Bel)$. From results on probabilistic transformations presented in [7] it can be concluded that, $Pl\_P(Bel) = Bel \oplus U_n$, for proof see [8]. From [3] we know that Pl_P commutes with $\oplus$, i.e. $Pl\_P(Bel_1 \oplus Bel_2) = Pl\_P(Bel_1) \oplus Pl\_P(Bel_2)$, thus we have homomorphism h for BFs on an n-element frame of discernment. To generalize all homomorphic properties of h we have also to verify a general versions of $h(0) = 0'$ and $h(0') = 0'$. It really holds true that $h(0, 0, ..., 0) = 0 \oplus U_n = (0, 0, ..., 0) \oplus (\frac{1}{n}, \frac{1}{n}, ..., \frac{1}{n}, 0, 0, ..., 0) = (\frac{1}{n}, \frac{1}{n}, ..., \frac{1}{n}, 0, 0, ..., 0) = U_n$. And similarly $h(U_n) = U_n \oplus U_n = U_n$. Hence the following theorem is proved. As there is no ordering of either BFs or elements of a frame of discernment, we cannot speak of ordered homomorphism as in two-element case.

Theorem 3 *The mapping $h(Bel) = Bel \oplus U_n = Pl\_P(Bel)$ is an homomorphism of an algebra of BFs on an n-element frame of discernment with the binary operation of Dempster's sum $\oplus$ and two nulary operations (constants) 0 and U_n.*

Thus, we can apply h with its homomorphic properties also in a general case. We have Bel and $h(Bel) = Pl\_P(Bel)$ which is BBF, i.e., BF which has upto n positive m-values (bbms). $h(Bel) = (h_1(Bel), h_2(Bel), ..., h_n(Bel), 0, 0, ...; 0)$; when interpreting $h(Bel)$ as a probability distribution on Ω, we have $h(Bel)(\omega_i) = h_i(Bel)$. We can use the following procedure to compute a related unique consonant BF Bel_0 to any $h(Bel)$.

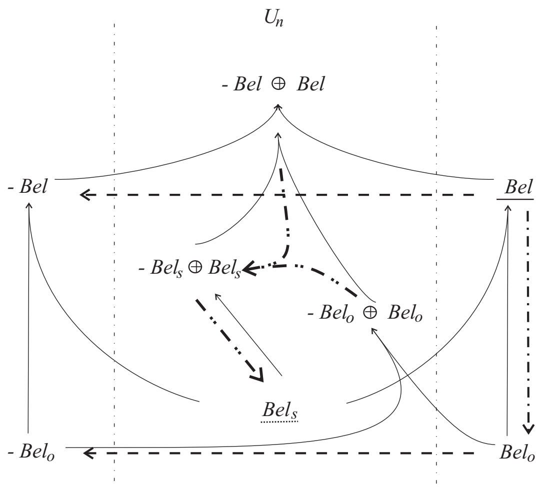

Figure 4: Schema of a decomposition of BF Bel.

Let there are k different values $h_i(Bel)$ for $i = 1, ..., n$, thus $1 \leq k \leq n$. According to this, we have splitting of the frame Ω into k disjoint subsets $\Omega = \Omega_1 \cup \Omega_2 \cup ... \cup \Omega_k$, such the the elements of the same subset have the same value $h(Bel)(\omega)$. Let $\Omega_1 = \{\omega_{11}, ..., \omega_{1j_1}\}$ be a set of elements of the frame with the highest m-value (bbm) ($h(Bel)(\omega_{11}) = h(Bel)(\omega_{12}) = ... = h(Bel)(\omega_{1j_1})$, where $1 \leq j_1 \leq n - k + 1$), and $\Omega_2 = \{\omega_{21}, ..., \omega_{2j_2}\}$ be a set of elements with the 2nd highest bbm ($h(Bel)(\omega_{21})$; $1 \leq j_2 \leq n - j_1 - k + 2$), then we define $m_w(\Omega_1) = h(Bel)(\omega_{11}) - h(Bel)(\omega_{21})$, further we define $m_w(\Omega_1 \cup \Omega_2) = h(Bel)(\omega_{21}) - h(Bel)(\omega_{31})$, where $h(Bel)(\omega_{31})$ is the 3rd largest m-value of $h(Bel)$. We continue similarly defining $m_w(\bigcup_{i=1}^{m} \Omega_i) = h(Bel)(\omega_{m1}) - h(Bel)(\omega_{(m+1)1})$, where $\Omega_i = \{\omega_{i1}, ..., \omega_{ij_i}\}$ is the set of elements with the i-th highest m-value of $h(Bel)$, until $m_w(\Omega) = h(Bel)(\omega_{k1})$ is defined, where $\Omega_k = \{\omega_{k1}, ..., \omega_{kj_k}\}$ is the set of elements with the least (possibly zero), m-value $h(Bel)(\omega_{k1})$, $j_k = n - \sum_{i=1}^{k-1} j_i$. $m_w(\bigcup_{i=1}^{m} \Omega_i) > 0$ for all $m < k$ because less value is always decreased, $m_w(\Omega_k) \geq 0$, $\sum_{m=1}^{k} m_w(\bigcup_{i=1}^{m} \Omega_i) = h(Bel)(\omega_{11}$. Then m_0 is a normalization of working bba m_w, thus focal elements of m_0 are nested and $Pl(\omega) = 1$ for $\omega \in \Omega_1$, hence Bel_0 is normalized consonant, i.e., non-conflicting BF. For detail and verification that, $Bel_0 \oplus U_n = h(Bel)$ and that $m_0 = (\frac{a-b}{1-b}, 0)$ is a special case of general m_0, see [10].

Finally, we can simplify the construction of Bel_0 in the following way: there is one normalization in computation of $Bel \oplus U_n = Pl\_P(Bel)$ and the following normalization in the transformation of m_w to m_0. Normalization commutes with the construction of m_w from $Pl\_P(Bel)$, thus when computing Bel_0, we can use $Pl(Bel)$ instead of $h(Bel) = Pl\_P(Bel)$ and apply only one normalization in the end, where normalization factor is the multiple of the original ones. Thus we obtain $m'_w(\{\omega_{11}, ..., \omega_{1j_1}\}) = Pl(Bel)(\omega_{11}) - Pl(Bel)(\omega_{21})$, etc. This computational simplification is important also from the theoretical point of view, because it removes Dempster's rule $\oplus$ hidden in h from the construction of Bel_0. Hence any Bel_0 has defined its non-conflicting part independently of any belief combination rule.

Lemma 2 *For any BF Bel defined on Ω_n there exists unique consonant BF Bel_0 such that,*

$$h(Bel_0 \oplus Bel_S) = h(Bel) \qquad (1)$$

for any BF Bel_S such that $Bel_S \oplus U_n = U_n$.

Proof. The existence follows the construction of Bel_0 when replacing (1) with $Bel_0 \oplus Bel_s \oplus U_n = Bel \oplus U_n$. For uniqueness we will also follow the construction of Bel_0: $h(Bel)$ is unique, thus also set of its m-values $h_i(Bel)$ is unique, k of them are different, $h_i(Bel)$ are real values from $[0, 1]$, thus their order is also unique, hence splitting of Ω into k disjoint subsets is unique as well, i.e. set of focal elements of m_w and m_0 is unique. Computation of differences is also unique thus we have unique m_w values and also their normalization m_0 values, hence m_0 is unique consonant bba such that $h(m_0) = h(Bel)$.
Futher it holds true that, $h(Bel_0 \oplus Bel_S) = h(Bel_0) \oplus h(Bel_S) = h(Bel) \oplus h(Bel_S) = h(Bel) \oplus U_n = h(Bel)$. $\square$

Let us notice, that the stronger statement for a general non-conflicting BFs does not hold true on Ω_n. There could be several different non-conflicting BFs Bel_i such that $h(Bel_i \oplus Bel_S) = h(Bel)$ for any indecisive BF B_S. See, the following example.

Example 1 *To BF* $Bel = (0.25, 0.175, 0.075, 0.35, 0.15, 0)$ *with* $h(Bel) = (0.25, 0.175, 0.075, 0.35, 0.15, 0) \oplus (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0, 0, 0) = (0.50, 0.35, 0.15, 0, 0, 0)$ *there are following non-conflicting BFs:* $Bel_0 = (0.3, 0, 0, 0.4, 0, 0; 0.3)$*,* $Bel_1 = (0, 0, 0, 0.7, 0.3, 0; 0)$*,* $Bel_2 = (0.2, 0, 0, 0.5, 0.1, 0; 0.2)$*;* $Pl_i(\{\omega_1\}) = 1$*, thus* Bel_i*s are all non-conflicting, we can simply verify that* $h(Bel_i) = h(Bel)$*, thus* $(Bel_i \oplus Bel_S) \oplus U_3 = Bel_i \oplus (Bel_S \oplus U_3) = Bel_i \oplus U_3 = h(Bel)$*.*

Let us turn our attention to $f(Bel)$ and $-Bel$ now. $f(a, b) = -(a, b) = (b, a)$ on Ω_2, thus we will try to generalize $-Bel$ to BFs on Ω_n now. We have nothing like S defined for BFs on Ω_n, thus we suppose $h(Bel \oplus -Bel) = U_n$ for $-Bel$. On Ω_2 it holds true that $-m(\{\omega_1\}) = m(\{\omega_2\}) = m(\Omega_2 \setminus \{\omega_1\})$, $-m(\{\omega_2\}) = m(\Omega_2 \setminus \{\omega_2\})$, and $-m(\Omega_2) = m(\Omega_2)$[5]. Unfortunately, the simple idea to define $-m$ as $-m(X) = m(\Omega_n \setminus X)$ does not work in general, not even for general consonant BFs, e.g., for $Bel = (0.5, 0, 0, 0.2, 0, 0; 0.3)$ and $\sim Bel = (0, 0, 0.2, 0, 0, 0.5; 0.3)$ we have $Bel \oplus \sim Bel = (\frac{15}{61}, \frac{10}{61}, \frac{6}{61}, \frac{6}{61}, \frac{0}{61}, \frac{15}{61}; \frac{9}{61})$, $(\frac{15}{61}, \frac{10}{61}, \frac{6}{61}, \frac{6}{61}, \frac{0}{61}, \frac{15}{61}; \frac{9}{61}) \oplus (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0, 0, 0; 0) = (\frac{30}{70}, \frac{40}{70}, \frac{30}{70}, 0, 0, 0; 0) = (\frac{3}{7}, \frac{4}{7}, \frac{3}{7}, 0, 0, 0; 0) \neq U_3$. Thus $h(Bel \oplus \sim Bel) \neq U_n$, hence $\sim Bel \neq -Bel$. The idea of complements $(\Omega \setminus X)$ works only in some special cases, e.g., for $(0.7, 0, 0, 0, 0, 0) \oplus (0, 0, 0, 0, 0, 0.7) = (21/51, 0, 0, 0, 0, \frac{21}{51}) \doteq (0.41, 0, 0, 0, 0, 0.41)$, $h(0.41, 0, 0, 0, 0, 0.41) = U_3$ on Ω_3 and for other simple support BFs in general.

To simplify the investigated situation, we will start with qBBFs on 3-element frame of discernment Ω_3 (i.e., with BFs such that $m(X) = 0$ for $|X| = 2$). The set of qBBFs on Ω_3 can be represented by a three dimensional triangle which simply generalizes the triangle of Dempster's pairs, see Figure 5. Unfortunately, the only consonant, i.e. non-conflicting, BFs are singleton simple support functions as $(a, 0, 0, 0, 0, 0; 1{-}a)$, thus only a small part of the triangle is mapped to non-conflicting BFs within the triangle (Bel_0 is outside of the triangle for a majority of qBBFs). Thus, this is not a good domain to search for $-Bel_0$.

Let us look at BBFs now, i.e. BFs as $(a, b, c, 0, 0, 0; 0) = (a, b, 1 - a - b, 0, 0, 0; 0)$. Let $-(a, b, 1-a-b, 0, 0, 0) = (x, y, 1-x-y, 0, 0, 0)$, thus

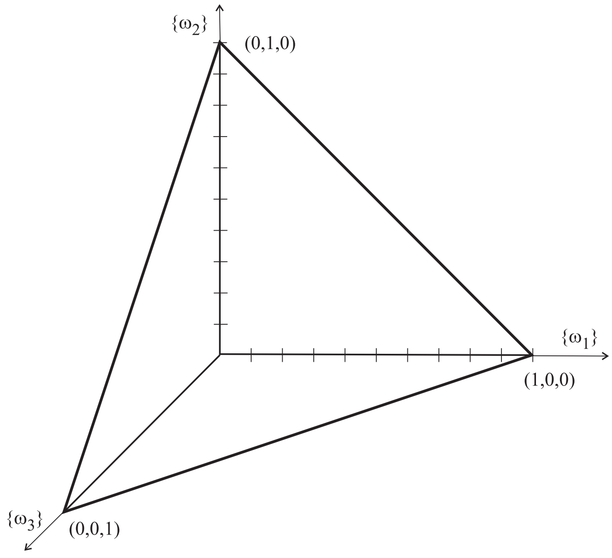

Figure 5: Quasi Bayesian BFs on 3-element frame Ω_3.

$-(a, b, 1-a-b, 0, 0, 0) \oplus (x, y, 1-x-y, 0, 0, 0) = U_3$ should hold true.

Thus $ax = by = (1 - a - b)(1 - x - y)$, $y = \frac{a}{b}x, (1 - x - y) = \frac{a}{1-a-b}x$, hence $1 - x - \frac{a}{b}x = \frac{a}{1-a-b}x$. Solving the previous equation we obtain $x = \frac{b(1-a-b)}{a+b-a^2-b^2-ab}$ and further $y = \frac{a(1-a-b)}{a+b-a^2-b^2-ab}$. Using $c = 1 - a - b$, we obtain $x = \frac{bc}{ab+ac+bc}$, $y = \frac{ac}{ab+ac+bc}$ and $1 - x - y = z = 1 - \frac{bc+ac}{ab+ac+bc} = \frac{ab}{ab+ac+bc}$. E.g. $(a, b, c, 0, 0, 0) = (0.5, 0.3, 0.2)$, $x = \frac{0.3 \cdot 0.2}{0.5 \cdot 0.3 + 0.5 \cdot 0.2 + 0.3 \cdot 0.2}$, $y = \frac{5 \cdot 2}{5 \cdot 3 + 5 \cdot 2 + 3 \cdot 2}$ $z = \frac{3 \cdot 2}{5 \cdot 3 + 5 \cdot 2 + 3 \cdot 2}$, thus $-(0.5, 0.3, 0.2, 0, 0, 0) = (\frac{6}{31}, \frac{10}{31}, \frac{15}{31}, 0, 0, 0)$.

Thus we have $-Bel$ for any BBF $(a, b, 1{-}a{-}b, 0, 0, 0)$ on Ω_3 such that $0 < a, b < 1, a + b < 1$.

Analogically to the case of Ω_3, we can generalize the $-Bel$ to BBFs on Ω_n, to BFs $(a_1, a_2, ..., a_n, 0, 0, ..., 0; 0)$ such that $0 < a_i < 1$, for $i = 1, ..., n$ and $a_n = 1 - \sum_{i=1}^{n-1} a_i$. Let us denote $-(a_1, a_2, ..., a_n, 0, 0, ..., 0; 0) = (x_1, x_2, ..., x_n, 0, 0, ..., 0; 0)$ (where $x_n = 1 - \sum_{i=1}^{n-1} x_i$), thus we obtain $x_1 = 1/(1 + \sum_{i=2}^{n} \frac{a_1}{a_i})$, $x_i = \frac{a_1}{a_i} x_1$, or similarly to x_1: $x_i = 1/(1 + \sum_{i \neq j} \frac{a_i}{a_j})$.

An alternative expression for x_i is $x_i = \frac{\prod_{i \neq j} a_j}{\sum_{k=1}^{n} \prod_{j \neq k} a_j}$, for detail see [10].

Lemma 3 *For any BBF* $(a_1, a_2, ..., a_n, 0, 0, ..., 0; 0)$ *such that,* $a_i > 0$ *for* $i = 1, ..., n$*, there exists uniquely defined* $-(a_1, a_2, ..., a_n, 0, 0, ..., 0; 0) = (x_1, x_2, ..., x_n, 0, 0, ..., 0; 0) = (1/(1 + \sum_{i=2}^{n} \frac{a_1}{a_i}), \frac{a_1}{a_2} x_1, \frac{a_1}{a_3} x_1, ..., \frac{a_1}{a_n} x_1, 0, 0, ..., 0; 0)$ *such that,*

$$(a_1, a_2, ..., a_n, 0, 0, ..., 0) \oplus -(a_1, a_2, ..., a_n, 0, 0, ..., 0) = U_n.$$

[5] Note that $-m(X)$ is an abbreviation for $(-m)(X)$, thus both $m(X)$ and $-m(X)$ may be positive in general. Specially $-m(\Omega_2)$ is an abbreviation for $(-m)(\Omega_2)$, thus $-m(\Omega_2) = m(\Omega_2)$, where both sides of the equation are positive in general.

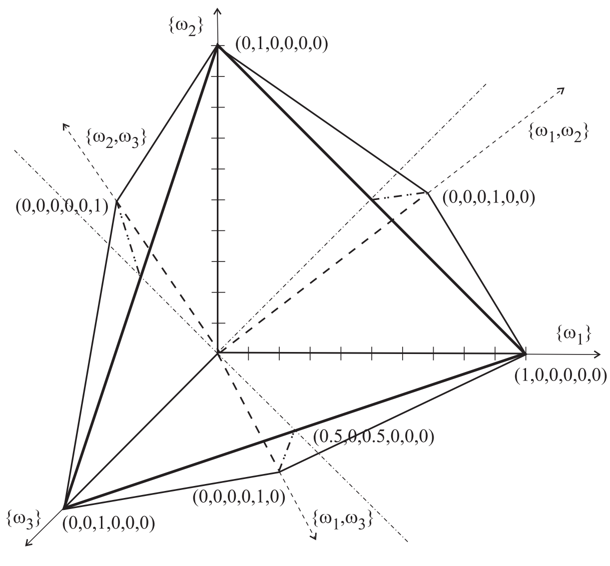

Figure 6: General BF on 3-element frame Ω_3.

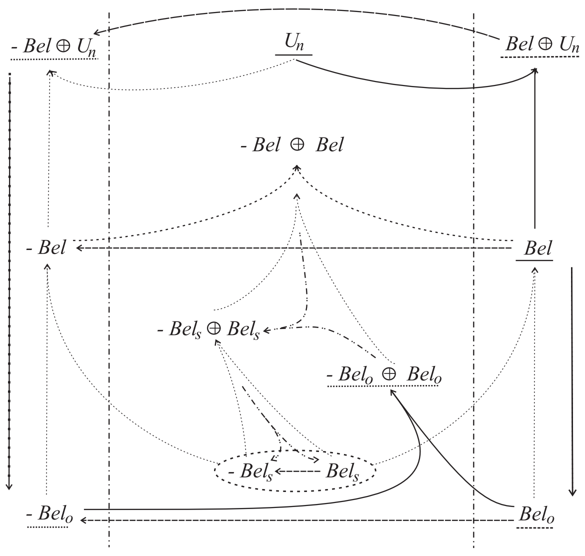

Figure 7: Detailed schema of a decomposition of BF Bel.

We have already observed, that $-Bel$ for a simple support function (SSF) is another SSF with a complementary focal element such that, $-m(\Omega_n \setminus X) = m(X)$; similarly we can define $-Bel$ also for simple support BBFs (i.e. categorical BBFs), see e.g., $-(1,0,0,0,0,0) = (0,0,0,0,0,1)$, but we have to notice that $(1,0,0,0,0,0) \oplus (0,0,0,0,0,1)$ is not defined (similarly to $(1,0) \oplus (0,1)$ on Ω_2). A definition of $-Bel$ for BBFs like $(a, 1-a, 0, 0, ..., 0)$ remains still open for more-element frames Ω_n, $n > 2$.

Summarising the previous results, we can step by step compute $h(Bel)$, $-h(Bel)$ and $(-h(Bel))_0$ from any Bel such that $Pl(\{\omega_i\}) > 0$ for all $\omega_i \in \Omega_n$, see Figure 7. Thus the following theorem holds true:

Theorem 4 *For any BF Bel defined on Ω_n there exists unique consonant BF Bel_0 such that,*

$$h(Bel_0 \oplus Bel_S) = h(Bel)$$

for any BF Bel_S such that $Bel_S \oplus U_n = U_n$. If for $h(Bel) = (h_1, h_2, ..., h_n, 0, 0, ..., 0)$ holds true that, $0 < h_i < 1$, then further exists unique BF $-Bel_0$ such that,

$$h(-Bel_0 \oplus Bel_S) = -h(Bel) \text{ and } h(Bel_0) \oplus -h(Bel_0) = U_n.$$

Proof. For existence and uniqueness of Bel_0 see Lemma 2. Existence of $-Bel_0$ follows its construction, $h(Bel)$ is unique according to its definition, for uniqueness of $-h(Bel)$ see Lemma 3 and final uniqueness of $-Bel_0$ follows Lemma 2 again. □

Corollary 1 *(i) For any consonant BF Bel such that $Pl(\{\omega_i\}) > 0$ there exist a unique BF $-Bel$; $-Bel$ is consonant in this case.*

(ii) There is one-to-one correspondence between Bayesian BFs and consonant BFs.

Proof. (i) Just take a consonant BF Bel, due to uniqueness of Bel_0 we have $Bel = Bel_0$, and also $-Bel = -Bel_0$. $Pl(\{\omega_i\}) > 0$ for all ω_i in the case of a consonant BF implies that $m(\Omega) > 0$, thus also $m_h(\{\omega_i\}) > 0$ for all ω_i, where $Bel_h = h(Bel) = Bel \oplus U_n$, thus we have $-Bel_h$ and $(-Bel_h)_0 = -Bel$; according its construction $-Bel$ is consonant and unique. If $Pl(\{\omega_i\}) = 0$ for some $\omega_i \in \Omega$, then $m(\Omega) = 0$, thus there exists ω_j such, that $m_h(\{\omega_j\}) = 0$, hence we have not defined either $-Bel_h$ or $-Bel$.
(ii) Taking any BBF Bel, we obtain unique consonant Bel_0; $h(Bel_0)$ is also unique. □

We observed that $-m(X) = m(\Omega \setminus X)$ for $X \subset \Omega$ and SSF m. We can verify that the definition of $-Bel$ using $-h(Bel)$ agree with this observation. E.g., $Bel = Bel_0 = (\frac{2}{3}, 0, 0, 0, 0, 0; \frac{1}{3})$, $h(Bel) = (\frac{3}{5}, \frac{1}{5}, \frac{1}{5}, 0, 0, 0; 0)$, $-h(Bel) = (\frac{1}{7}, \frac{3}{7}, \frac{3}{7}, 0, 0, 0; 0)$, and $-Bel = (-h(Bel))_0 = (0, 0, 0, 0, 0, \frac{2}{3}; \frac{1}{3})$. In general we have $m(X) = a$ and $m(\Omega) = 1 - a$, where $|X| = k, |\Omega| = n$. Thus $h(m)(\omega_i) = \frac{a+(1-a)}{k(a+(1-a))+(n-k)(1-a)} = \frac{1}{n-(n-k)a}$ for $\omega_i \in X$ and $h(m)(\omega_j) = \frac{1-a}{k(a+(1-a))+(n-k)(1-a)} = \frac{1-a}{n-(n-k)a}$ for $\omega_j \in \Omega \setminus X$. Further, $-h(m)(\omega_i) = \frac{1^{k-1}(1-a)^{n-k}}{k \cdot 1^{k-1}(1-a)^{n-k}+(n-k)1^k(1-a)^{n-k-1}} = \frac{1-a}{n-ka}$, $-h(m)(\omega_j) = \frac{1^k(1-a)^{n-k-1}}{k \cdot 1^{k-1}(1-a)^{n-k}+(n-k)1^k(1-a)^{n-k-1}} = \frac{1}{n-ka}$, hence $-m(\Omega \setminus X) = (-h(m))_0(\Omega \setminus X) = (-h(m))(\omega_j) - (-h(m))(\omega_i) = \frac{\frac{1}{n-ka} - \frac{1-a}{n-ka}}{\frac{1}{n-ka} - \frac{1-a}{n-ka} + \frac{1-a}{n-ka}} = a$, and $-m(\Omega) = (-h(m))_0(\Omega) = (-h(m))(\omega_i) =$

$\frac{\frac{1-a}{n-ka}}{\frac{1}{n-ka}-\frac{1-a}{n-ka}+\frac{1-a}{n-ka}} = 1 - a$. Thus really $-m(\Omega \setminus X) = m(X)$ and $-m(\Omega) = m(\Omega)$. □

For completion of the diagram in Figure 7, we need a definition of $-Bel$ for general BFs on Ω to compute $Bel \oplus -Bel$ and analysis of indecisive BFs (i.e. BFs Bel such that, $h(Bel) = U_n$) to compute $Bel_S \oplus -Bel_S$ and resulting Bel_S and specify conditions under which Bel_S is defined and unique. Hence an algebraic analysis of BFs on a general finite frame of discernment is required.

5 Comments on other belief combination rules

There arises an interesting question about similar kind of decomposition of belief functions with another combination rules.

As it was already mentioned, the non-conflicting part Bel_0 of a belief function Bel defined above is independent from Dempster's rule of combination, as we can use the representation of homomorphism h using normalized plausibility of singletons $Pl\_P(Bel)$ instead of the original $h(Bel) = Bel \oplus U_n$. Thus Bel_0 can be computed without any relation to Dempster's rule and $Pl\_P(Bel_0) = Pl\_P(Bel)$ independently from any combination rule.

On the other hand $Pl\_P(Bel) \neq Bel_0 \textcircled{Y} U_n$, $Pl\_P(Bel) \neq Bel_0 \textcircled{D} U_n$, $Pl\_P(Bel) \neq Bel_0 \textcircled{\cup} U_n$, see Example 2. Even $Pl\_P(Bel) \neq Pl\_P(Bel_0 \circledast U_n)$, where $\circledast$ is either $\textcircled{Y}$, $\textcircled{D}$, $\textcircled{\cup}$ or some other combination rule. The equality holds true only for Dempster's rule: $Pl\_P(Bel) = Bel_0 \oplus U_n$; in the case of un-normalized conjunctive rule $\textcircled{\cap}$ we can apply additional normalization to obtain the equality, thus we have normalized conjuntive rule, i.e., Dempster's rule $\oplus$ again.

Example 2 *Let us take* $Bel = (0.3, 0.2, 0.1, 0.2, 0.1, 0.0; 0.1)$, *thus there is* $Pl = (0.7, 0.5, 0.3, 0.9, 0.8, 0.7; 1.0)$, $Pl\_P(Bel) = (\frac{7}{15}, \frac{5}{15}, \frac{3}{15})$, *and* $Bel_0 = (\frac{2}{7}, 0, 0, \frac{2}{7}, 0, 0; \frac{3}{7})$. *Hence we obtain* $(\frac{2}{7}, 0, 0, \frac{2}{7}, 0, 0; \frac{3}{7}) \oplus (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0, 0, 0; 0) = (\frac{7}{15}, \frac{5}{15}, \frac{3}{15}, 0, 0, 0; 0)$*; but* $(\frac{2}{7}, 0, 0, \frac{2}{7}, 0, 0; \frac{3}{7}) \textcircled{Y} (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0, 0, 0; 0) = (\frac{7}{21}, \frac{5}{21}, \frac{3}{21}, 0, 0, 0; \frac{6}{21}) \neq (\frac{7}{15}, \frac{5}{15}, \frac{3}{15}, 0, 0, 0; 0)$, $(\frac{2}{7}, 0, 0, \frac{2}{7}, 0, 0; \frac{3}{7}) \textcircled{D} (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0, 0, 0; 0) = (\frac{7}{21}, \frac{5}{21}, \frac{3}{21}, \frac{2}{21}, \frac{2}{21}, 0; \frac{2}{21}) \neq (\frac{7}{21}, \frac{5}{21}, \frac{3}{15}, 0, 0, 0; 0)$, *and similarly* $(\frac{2}{7}, 0, 0, \frac{2}{7}, 0, 0; \frac{3}{7}) \textcircled{\cup} (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0, 0, 0; 0) = (\frac{2}{21}, 0, 0, \frac{6}{21}, \frac{2}{21}, 0; \frac{11}{21}) \neq (\frac{7}{15}, \frac{5}{15}, \frac{3}{15}, 0, 0, 0; 0)$. *Further* $Pl\_P(\frac{7}{21}, \frac{5}{21}, \frac{3}{21}, 0, 0, 0; \frac{6}{21}) = (\frac{13}{33}, \frac{11}{33}, \frac{9}{33}) \neq (\frac{7}{15}, \frac{5}{15}, \frac{3}{15})$, $Pl\_P(\frac{7}{21}, \frac{5}{21}, \frac{3}{15}, \frac{2}{21}, \frac{2}{21}, 0; \frac{2}{21}) = (\frac{13}{29}, \frac{9}{29}, \frac{7}{29}) \neq (\frac{7}{15}, \frac{5}{15}, \frac{3}{15})$, *and* $Pl\_P((\frac{2}{21}, 0, 0, \frac{6}{21}, \frac{2}{21}, 0; \frac{11}{21}) = (\frac{21}{51}, \frac{17}{51}, \frac{13}{51}) \neq (\frac{7}{15}, \frac{5}{15}, \frac{3}{15})$.

Nevertheless, if there is a couple of homomorphisms for any combination rule $\circledast$ analogic to morphisms f and h from Dempster's semigroup, then there exists an analogy of Bel_0 also for the combination rule $\circledast$.

When expressing h using $Pl\_P(Bel)$ there arises another interesting question about similar kind of non-conflicting part and decomposition of belief functions using a different probabilistic transformations.

Considering Smets' pignistic transformation $BetT$ for computing pignistic probability $BetP$ we obtain non-conflicting BF $Bel_{0\text{–}BetP}$, where $m_{w\text{–}BetP}(\bigcup_{i=1}^{m} \Omega_i) = |\bigcup_{i=1}^{m} \Omega_i|(h(Bel)(\omega_{m1}) - h(Bel)(\omega_{(m+1)1}))$, which is normalized, hence $m_{0\text{–}BetP} = m_{w\text{–}BetP}$. $BetT$ does not commute either with Dempster's rule nor with other rules defined for belief combination, thus we cannot use $Bel_{0\text{–}BetP}$ for decomposition of belief functions to conflicting and non-conflicting parts. For counter-examples see [10].

The most perspective pignistic transformation is normalized belief of singletons Bel_P which is compatible with disjunctive rule of combination [7], unfortunately, the reverse transformation maps any Bel and $Bel\_P(Bel)$ to the vacuous belief function $0 = (0, 0, ..., 0; 1)$, which is really non-conflicting, but it does not reprezent non-conflicting part of the belief function Bel. In this case it represents zero conflicting part, as the disjunctive rule is completely non-conflicting; thus it holds true $Bel = Bel \textcircled{\cup} 0$, where Bel is trivial 'disjunctive non-conflicting' part of itself and 0 is trivial 'disjunctive conflicting' part of any BF Bel.

Moreover, it is possible to show that there is no similar decomposition of belief functions for $\textcircled{Y}$, $\textcircled{D}$, $\textcircled{\cup}$ and a for a series of other combination rules, see [10]. Any Bayesian BF serves as counter-example there.

6 Conclusion

Decomposition of a belief function (BF) defined on a two-element frame of discernment to Dempster's sum of its unique non-conflicting and unique indecisive conflicting part is defined and presented here.

Homomorphic properties of mapping $h(Bel) = Bel \oplus U_n$ which corresponds to normalized plausibility of singletons were verified for BFs defined on a general finite frame of discernment. $-Bel$ was generalized to Bayesian BFs and for consonant BFs on a general n-element frame.

Further a unique consonant non-conflicting part Bel_0 of a general BF Bel on a finite frame was defined. For specification of a corresponding conflicting part of Bel and its uniqueness/existence properties, an algebraic analysis of BFs on a general finite frame of discernment is required.

The presented topic is finally discused also from the point of view of alternative rules of combination and alternative probabilistic transformations.

The presented results improve general understanding of belief functions and their combination, especially in conflicting cases. They can be used as one of corner-stones to further study of conflicts between belief functions.

Acknowledgments

This research is supported by the grant P202/10/1826 of the Grant Agency of the Czech Republic.

Partial support by the Institutional Research Plan AV0Z10300504 "Computer Science for the Information Society: Models, Algorithms, Applications" is also acknowledged.

The author is grateful to Eva Pospíšilová for creation of useful and illuminative figures.

References

[1] R. G. Almond: Graphical Belief Modeling. Chapman & Hall, London, 1995.

[2] A. Ayoun, Ph. Smets: Data association in multi-target detection using the transferable belief model. *International Journal of Intelligent Systems* **16** (10): 1167–1182, 2001.

[3] B. R. Cobb, P. P. Shenoy: A Comparison of Methods for Transforming Belief Functions Models to Probability Models. In: Nielsen, T. D., Zhang, N. L. (eds.) ECSQARU 2003. LNAI 2711: 255–266. Springer, Heidelberg, 2003.

[4] M. Daniel: Algebraic structures related to Dempster-Shafer theory. In B. Bouchon-Meunier, R. R. Yager, and L. A. Zadeh (eds.), *Advances in Intelligent Computing - IPMU'94*. LNCS 945: 51–61. Springer-Verlag, Berlin Heidelberg, 1995.

[5] M. Daniel: Distribution of Contradictive Belief Masses in Combination of Belief Functions. In: B. Bouchon-Meunier, R. R. Yager, L. A. Zadeh (eds.) *Information, Uncertainty and Fusion*, pp. 431–446. Kluwer Acad. Publ., Boston, 2000.

[6] M. Daniel: Algebraic Structures Related to the Combination of Belief Functions. *Scientiae Mathematicae Japonicae* **60** (2): 245–255, 2004. *Sci. Math. Jap. Online* **10**: 501–511, 2004.

[7] M. Daniel: Probabilistic Transformations of Belief Functions. In: Godo, L. (ed.) Symbolic and Quantitative Approaches to Reasoning with Uncertainty (ECSQARU 2005). LNAI 3571: 539–551. Springer, Heidelberg, 2005.

[8] M. Daniel: New Approach to Conflicts within and between Belief Functions. Technical report V-1062, ICS AS CR, Prague, 2009.

[9] M. Daniel: Conflicts within and between Belief Functions. In: E. Hüllermeier, R. Kruse, E. Hoffmann (eds.) *IPMU 2010*. LNAI 6178: 696–705. Springer-Verlag, Berlin Heidelberg, 2010.

[10] M. Daniel: *Conflicts of Belief Functions*. Technical report V-1108, ICS AS CR, Prague, 2011.

[11] D. Dubois, H. Prade: Representation an combination of uncertainty with belief functions and possibility measures. *Computational Intelligence*, **4**: 244–264, 1988.

[12] D. Dubois, H. Prade: Consonant Approximations of Belief Functions. *International Journal of Approximate Reasoning* **4**: 419–449, 1990.

[13] R. Haenni: Aggregating Referee Scores: an Algebraic Approach. *COMSOC'08, 2nd International Workshop on Computational Social Choice*, Liverpool, UK, 2008.

[14] P. Hájek, T. Havránek, R. Jiroušek: *Uncertain Information Processing in Expert Systems*. CRC Press, Boca Raton, Florida, 1992.

[15] P. Hájek, J. J. Valdés: Generalized algebraic foundations of uncertainty processing in rule-based expert systems (dempsteroids). *Computers and Artif. Intell.* **10** (1): 29–42, 1991.

[16] W. Liu: Analysing the degree of conflict among belief functions. Artificial Intelligence **170**: 909–924, 2006.

[17] G. Shafer: *A Mathematical Theory of Evidence*. Princeton University Press, New Jersey, 1976.

[18] Ph. Smets: The combination of evidence in the transferable belief model. IEEE-Pattern analysis and Machine Intelligence **12**: 447–458, 1990.

[19] Ph. Smets: Analyzing the combination of conflicting belief functions. *Information Fusion* **8**: 387–412, 2007.

[20] J. J. Valdés: *Algebraic and logical foundations of uncertainty processing in rule-based expert systems of Artificial Intelligence*. PhD Thesis, Czechoslovak Academy of Sci., Prague, 1987.

[21] R. R. Yager: On the Demspter-Shafer framework and new combination rules. *Information Sciences*, **41**: 93–138, 1987.