Overcoming some limitations of imprecise reliability models

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Abstract

The application of imprecise reliability models is often hindered by the rapid growth in imprecision that occurs when many components constitute a system and by the fact that time to failure is bounded from above. The latter results in the necessity to explicitly introduce an upper bound on time to failure which is in reality a rather arbitrary value. The practical meaning of the models of this kind is brought to question. We suggest an approach that overcomes the issue of having to impose an upper bound on time to failure and makes the calculated lower and upper reliability measures more precise. The main assumption consists in that failure rate is bounded. Lagrange method is used to solve the non-linear program. Finally, an example is provided.

Keywords. Imprecise reliability, variational calculus, bounded failure rate.

1 Introduction

The appropriate incorporation of uncertainty into reliability and risk analyses is a topic of importance and widespread interest. Perhaps the most widely recognised distinction in uncertainty types is between aleatory and epistemic uncertainty and the presence of these two in the analyses of complex systems is a challenge systems analysts face. To address it, a number of mathematical structures able to capture the both types have been developed. The reader can find good overviews of the methods of uncertainty representation in different sources, for example, in [1] – [4]. Some of the mathematical structures are based on the two simple notions: interval-valued probabilities and imprecisely specified probability distributions. These structures are interval probability, probability bound analysis, Dempster-Shafer theory, robust Bayes methods, and the theory of imprecise probabilities that can be considered as the most general approach. The theory of imprecise probabilities, as it was introduced in [1] and [5], has served as the theoretical basis for generalising a large number of reliability models to imprecise probabilities. For a brief overview see [6]. More specifically, the reliability models of non-reparable systems of general structures (series, parallel and complex connection) generalised to imprecise probabilities are presented in [7], generalised discrete Markov chains used to model repairable systems are described in [8] and [9], stress-strength models for structural reliability are reported in [10] and [11]. The theory of imprecise probabilities has been applied to other important issues for reliability and risk analyses like aggregation of imprecise data having' different degrees of confidence to different pieces of evidence, expert judgement elicitation procedures, and decision making based on imprecise probabilities.

In spite of the seemingly rich arsenal of applied models built on imprecise statistical reasoning, they are nevertheless hesitantly used in practice and remain firmly in the academic realm. Do they lack adequate promotion by their practitioners, or are there other primary obstacles that prevent them from being widely applied? In [12] the authors' belief was that the main obstacle to the practical application of this knowledge is a tangible imprecision in lower and upper probability bounds constructed from a set of imprecise probabilistic pieces of evidence or/and the rapid growth in imprecision that occurs when intervals are propagated through mathematical models. The main cause in mathematical terms of the tangible imprecision was arguably identified as lying in the main mechanism of constructing coherent imprecise probability measures, which was originally called by Walley natural extension [1], and which in fact is a linear program. The crux of this linear program is that the solutions obtained are defined on the family of degenerate probability distributions[1], which are included on equal footing in the set of all admissible probability distributions over which the solution is sought. As proven in [13], solving this optimisation problem on the set of all admissible probability distributions gives the same solution as that obtained on only the set of degenerate distributions. This would simply be

[1] The probability distribution of a continuous random variable is referred to as degenerate if the probability masses are concentrated in a finite number of points belonging to the continuous set of possible states.

mathematical subtlety – that is, of little interest to practitioners – if it did not give us a clue to deriving more precise previsions of interest for continuous random variables. For some variables it is often not realistic to assume that the probability masses are concentrated in a few points as opposed to being continuously distributed over the set of possible outcomes. In reliability applications probability masses of time to failure cannot (except for very special cases) concentrate in a very few points of the positive real line. Ignoring this fact is one of the causes (we hold it to be the root cause) of high imprecision in reliability as well as in other applications. Or at least this is where some improvements are possible.

Several attempts have been undertaken to introduce some extra judgements to the set of constraints of the natural extension to limit the set of admissible probability distributions on which a solution is sought. That is, the desire is to remove from the admissible set the distributions that are obviously do not provide a reasonable model of the underlying random values like time to failure.

An attempt to mitigate the influence of degenerate probability distributions on the solutions was undertaken in [14]. No significant effect was obtained through the introduction of judgements on the skewness and unimodality of the distributions as, in this case, the peaks of degenerate distributions simply become repositioned and probability masses become redistributed among the peaks. The nature of the distributions defining the solutions remains unchanged.

Another approach was suggested in [15]. It consists in employing the calculus of variations to solve the optimisation problems instead of attempting to solve them with linear programming techniques. As it was demonstrated in [15] and then in [12] and [16] this way enables us to utilise a broader spectrum of statistical judgements, which results in tighter bounds on probability measures. The introduction of direct constraints on probability distributions like an upper bound on a probability density function (pdf) or/and on the absolute value of its derivative turned to be especially efficient. This type of constraints is not possible to utilise if the conventional natural extension in the form of a linear program is used as a tool for construction of imprecise probability measures. Direct constraints on pdfs make the problem nonlinear that can be solved with variational calculus. The direct constraints result in good improvements in precision so that we can see room for even better improvements.

Despite the obvious improvements in the precision of the constructed measures there is yet one more obstacle on the way of applying the theory of imprecise probabilities to reliability calculations. This obstacle stems from the underlying constraint imposed on the values of random variables. The random variables are bounded and this feature has a pernicious consequence on imprecise reliability models. This consequence consists in having an upper bound on time to failure explicitly present in the reliability models. (The lower bound is present too but since it is equal to zero, seemingly it is not part of the models.) Why the consequence is so harmful? This is because the upper bound on time to failure of any systems cannot be known. That is to say, the imposed necessity to choose this bound makes the reliability measures rather arbitrary values, as the upper bound is not known. The only non-arbitrary and true assertion about the sample space of time to failure is that it stretches from zero to infinity. All conventional reliability models reside in this presupposition.

In this paper we continue to use the calculus of variations for constructing imprecise probability measures and we introduce constraints on failure rate. It has a double effect: better precision in the results and avoidance of the necessity to have the upper bound on time to failure.

2 Exhibiting imprecise reliability models with the troublesome parameter

Let us look at several reliability models generalised to imprecise probabilities. The notations used are the following: $\underline{a}_i$ and $\overline{a}_i$ are a lower and upper m-th moments of time to failure of an i-th component for $i \leq n$, $\underline{A}$ and $\overline{A}$ are a lower and upper m-th moments of a system compounded of n components, and T is an upper bound of time to failure that is assumed the same for all components.

For a system with independent components connected in series from the reliability point of view the following results are valid [7]:

$$\underline{A} = \frac{1}{(T^{n-1})^m} \prod_{i=1}^{n} \underline{a}_i \,, \qquad \overline{A} = \min_{i=1,\dots,n} \overline{a}_i$$

If the components are connected in parallel, then [7]

$$\underline{A} = \max_{i=1,\dots,n} \underline{a}_i \,, \qquad \overline{A} = T - T \prod_{i=1}^{n} \left(1 - \frac{\overline{a}_i}{T}\right)$$

Consider a couple of more examples. Let K is an upper bound of the pdf of time to failure of a component and this is the only reliability data available. Then we have the following results for the mean time to failure $M(t)$ [12]:

$$\underline{M}(t) = \frac{1}{2K} \,, \qquad \overline{M}(t) = T - \frac{1}{2K}$$

If in addition to K a bound on the absolute value of the pdf's derivative L is known, then [16]

$$\underline{M}(t) = \frac{1}{2K} + \frac{K}{2L}, \qquad \overline{M}(t) = T - \frac{1}{2K} - \frac{K}{2L}$$

As seen from the above expressions, one of the bounds of the expected values is explicitly dependent on the upper bound of time to failure T. Assuming that $T \to \infty$ gives us a very imprecise result that in many cases is practically useless. The two interrelated issues - high imprecision and dependence on the upper bound of time to failure – have motivated us to attempt to find a better solution.

The following section suggests a new problem statement that - as it will be demonstrated further in this paper - results in improved solutions.

3 Problem statement

Let us formulate first a rather general problem of computing bounds $\underline{M}$ and $\overline{M}$ on the expected value of an arbitrary function $g(x)$ given the upper, $\overline{f}_i = \overline{M}(f_i(t))$, and lower, $\underline{f}_i = \underline{M}(f_i(t))$, bounds of the expected values of other arbitrary functions $f_i(t)$, $i \le n$. As a particular case, the expected values can be known precisely meaning that the bounds are equal to each other. If $f_i(t) = t$, the expected value is the first moment. If $f_i(t) = t^2$, the expected value is the second moment, etc. In case $f_i(t) = I_{[t_1,t_2]}(t)$, where $I_{[t_1,t_2]}(t)$ is an indicator function equal to 1 when $t \in [t_1, t_2]$, and equal to 0 otherwise, the expected value is the probability $Pr(t \in [t_1, t_2])$.

The problem is stated as follows:

$$\left.\begin{aligned} \underline{M}(g) &= \inf_{\{\rho(x)\}} \int_0^T g(x)\rho(x)dx \\ \overline{M}(g) &= \sup_{\{\rho(x)\}} \int_0^T g(x)\rho(x)dx \end{aligned}\right\} \qquad (1)$$

subject to

$$\left.\begin{aligned} &\underline{f}_i \le \int_0^T f_i(x)\rho(x)dx \le \overline{f}_i, \quad i = 1,2,\ldots,n \\ &\rho(x) \ge 0, \text{ and } \int_0^T \rho(x)dx = 1 \end{aligned}\right\} \qquad (2)$$

where $\rho(x)$ is the pdf of a random variable x defined on $[0,T]$. Here the inf and sup are taken over the set $\{\rho(x)\}$ of all pdfs matching constraints (2). That is, each constraint in (2) is associated with a subset of $\{\rho(x)\}$, and the intersection of those subsets, if not empty, defines the solutions of the optimization problems (1)-(2). If some of the subsets of $\{\rho(x)\}$ become disjoint, the solution does not exist. It should be noted that problems (1)-(2) are linear and the dual optimization problems can be written for them. The primal optimisation problems (1)-(2) and their duals have served as the key tools to derive a number of imprecise reliability models (see, for example, [7], [8] and [14]). The results were explicitly dependent on the upper bound, T, imposed on the random variable time to failure, as it was demonstrated in the previous section.

This is namely problems (1)-(2) the solutions to which are defined on the family of degenerate probability distributions [13]. This finding was a point of departure for introducing constraints that rule out the degenerate distribution from the set of admissible ones. Being guided by this finding, tighter bounds for probability measures have been derived for several problem statements [12], [15], [16]. In this paper we seek to solve the more ambitious problem: obtaining tighter bounds for a constructed probability measure of interest and getting rid of the need to impose an upper bound, T, on time to failure.

Now we introduce some new constraints and reformulate problems (1)-(2). In the following we will think of the random variable t as time to failure.

The cumulative distribution function of time to failure takes the form

$$F(t) = \int_0^t \rho(x)dx$$

and the reliability function is $P(t) = 1 - F(t)$. According to its definition (see, for example, [17]) the failure rate is

$$\lambda(t) = \frac{\rho(t)}{P(t)},$$

from which $P(t) = exp\left(-\int_0^t \lambda(x)dx\right)$.

Denote $\int_0^t \lambda(t)dt = y(t)$, then $\lambda(t) = \frac{dy(t)}{dt} = y'(t)$

Based on the above formulas and introduced notation the expression for the pdf, $\rho(t)$, appears as follows

$$\rho(t) = P(t)\lambda(t) = y'(t)exp(-y(t)).$$

Assuming that the failure rate is bounded from below and above by $\underline{\lambda}$ and $\overline{\lambda}$, that is $\underline{\lambda} \le \lambda(t) = y'(t) \le \overline{\lambda}$ and considering the lower, $\underline{f_i}$, and upper, $\overline{f_i}$, bound on the expected value of random variable $f_i(t)$ known, the following optimisation problem can be formulated

$$\left.\begin{aligned} \underline{M}(g) &= \inf_{\{y(t)\}} \int_0^T g(t) y'(t) \exp(-y(t)) dt \\ \overline{M}(g) &= \sup_{\{y(t)\}} \int_0^T g(t) y'(t) \exp(-y(t)) dt \end{aligned}\right\} \quad (3)$$

subject to

$$\underline{f_i} \le \int_0^T f_i(t) y'(t) \exp(-y(t)) dt \le \overline{f_i}, \quad i \le n, \quad (4)$$

$$\int_0^T y'(t) \exp(-y(t)) dt = 1 \quad (5)$$

$$\underline{\lambda} \le y'(t) \le \overline{\lambda} \quad (6)$$

Problems (3)-(6) are nonlinear and in order to solve them we suggest employing the calculus of variations as it was done in [12], [15], and [16].

4 Solving the problem with the calculus of variations

Problems similar to (3)-(6) have to be modified slightly to make them amenable to the calculus of variations. The constraint $\underline{\lambda} \le y'(t) \le \overline{\lambda}$ can be rewritten as follows:

$$\begin{aligned} y'(t) - u^2(t) &= \underline{\lambda}, \\ y'(t) + v^2(t) &= \overline{\lambda}. \end{aligned} \quad (7)$$

Here $u(t)$, $v(t)$ are unknown real-valued functions.

The solution of problems (3) subject to constraints (4), (5) and (7) is based on the following theorem

Theorem. *If for any interval $\alpha \le t \le \beta$, $0 \le \alpha < \beta \le T$ and for any $h_0, h_1, \ldots, h_n \in \mathbf{R}$ it holds that*

$$g(t) \ne h_0 + \sum_{i=1}^{n} h_i f_i(t),$$

then the failure rate $\lambda(t) = y'(t)$, on which inf and sup are attained in problems (3) subject to constraints (4), (5) and (7), is a step-wise function which is equal either to $\underline{\lambda}$ or to $\overline{\lambda}$.

The proof of this theorem is given in the Appendix and the meaning of it is that $\lambda(t)$ cannot take any other values between $\underline{\lambda}$ and $\overline{\lambda}$ but only either $\underline{\lambda}$ or $\overline{\lambda}$. This statement has a direct influence on the pdf, $\rho(t)$, on which inf and sup are attained in problems (3). That is, the pdf consists of the pieces $\underline{\rho}(t) = p(t_0, \ldots, t_i) \cdot \underline{\lambda} \cdot exp(-\underline{\lambda}(t - t_i))$, $t \ge t_i$ and $\overline{\rho}(t) = p(t_0, \ldots, t_{i-1}) \cdot \overline{\lambda} \cdot exp(-\overline{\lambda}(t - t_{i-1}))$, $t \ge t_{i-1}$ that switch at some instances $t_1, t_2, \ldots, t_i$. The term $p(t_0, \ldots, t_i)$ is interpreted as the probability of being free of failure until time instant t_i. The correspondence between $\underline{\lambda}$, $\overline{\lambda}$ and optimizing $\rho(t)$ is shown in Fig. 1.

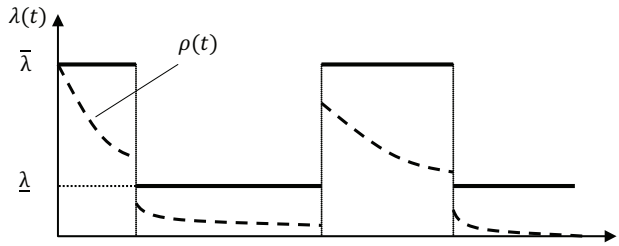

Figure 1. Optimizing pdf, $\boldsymbol{\rho(t)}$, and connected to it $\underline{\boldsymbol{\lambda}}$ and $\overline{\boldsymbol{\lambda}}$

Noticeable, the distribution of probability masses over time tends to zero when time tends to infinity. This in fact means that the very strong limiting requirement of imprecise probability theory that the random variable must be bounded is no longer valid and the "troublesome parameter" will not enter the expressions for reliability measures. It will be demonstrated in an example below.

As now the optimizing pdf is known (except for t_i) we can return to optimization problems (1)-(2) where $\rho(t)$ explicitly appears in the formulas. That what is not known now is the instances t_i when $\lambda(t)$, and consequently, $\rho(t)$ switch from one to the other value.

Assume that the optimal failure rate $\lambda(t)$ commutes $2m$ times between $\underline{\lambda}$ and $\overline{\lambda}$. That is,

$[0, t_1)$, $[t_2, t_3)$, $[t_4, t_5)$,...,$[t_{2j}, t_{2j+1})$,...

are intervals of time in which $\lambda(t) = \overline{\lambda}$. Similarly,

$[t_1, t_2)$, $[t_3, t_4)$, $[t_5, t_6)$,...,$[t_{2j+1}, t_{2j+2})$,... are the intervals on which $\lambda(t) = \underline{\lambda}$, $j \le m$.

Note that if $m = 0$, we have 2 intervals: one with the failure rate equal to $\overline{\lambda}$ and other with the failure rate equal to $\underline{\lambda}$. There may be some cases for which the optimizing failure rate for the whole time interval $[0, T]$ is constant and equal either to $\overline{\lambda}$ or $\underline{\lambda}$.

The formula $\int_0^T f_i(x)\rho(x)dx$ for the expected value appearing in the constraints (2) as well as $\int_0^T g(t)\rho(t)dt$ appearing in (1) can now be rewritten

$$\Phi_i = \int_0^T f_i(x)\rho(x)dx =$$

$$\overline{\lambda}\left(\sum_{j=0}^{m} p(t_0,...,t_{2j}) \int_{t_{2j}}^{t_{2j+1}} f_i(t)\exp\left(-\overline{\lambda}(t-t_{2j})\right)dt\right) +$$

$$\underline{\lambda}\left(\sum_{j=1}^{m} p(t_0,...,t_{2j+1}) \int_{t_{2j+1}}^{t_{2j+2}} f_i(t)\exp\left(-\underline{\lambda}(t-t_{2j+1})\right)dt\right).$$

$$G = \int_0^T g(t)\rho(t)dt =$$

$$\overline{\lambda}\left(\sum_{j=0}^{m} p(t_0,...,t_{2j}) \int_{t_{2j}}^{t_{2j+1}} g(t)\exp\left(-\overline{\lambda}(t-t_{2j})\right)dt\right) +$$

$$\underline{\lambda}\left(\sum_{j=1}^{m} p(t_0,...,t_{2j+1}) \int_{t_{2j+1}}^{t_{2j+2}} g(t)\exp\left(-\underline{\lambda}(t-t_{2j+1})\right)dt\right).$$

$$R = \int_0^T \rho(t)dt = \overline{\lambda}\left(\sum_{j=0}^{m} p(t_0,...,t_{2j}) \int_{t_{2j}}^{t_{2j+1}} \exp\left(-\overline{\lambda}(t-t_{2j})\right)dt\right) +$$

$$\underline{\lambda}\left(\sum_{j=1}^{m} p(t_0,...,t_{2j+1}) \int_{t_{2j+1}}^{t_{2j+2}} \exp\left(-\underline{\lambda}(t-t_{2j+1})\right)dt\right) =$$

$$\sum_{j=0}^{m} p(t_0,...,t_{2j})\cdot\left(1-\exp(-\overline{\lambda}(t_{2j+1}-t_{2j})\right) +$$

$$\sum_{j=0}^{m} p(t_0,...,t_{2j+1})\cdot\left(1-\exp(-\underline{\lambda}(t_{2j+2}-t_{2j+1})\right).$$

Finally, the reformulated problem statement is as follows:

$$\min_{t_1,t_2,...,t_{2m+2}} G(t_1,t_2,...,t_{2m+2}) \text{ and}$$

$$\max_{t_1,t_2,...,t_{2m+2}} G(t_1,t_2,...,t_{2m+2})$$

subject to constraints

$$\underline{a}_i \le \Phi_i(t_1,t_2,...,t_{2m+2}) \le \overline{a}_i,\ \ i \le n, \text{ and}$$

$$R(t_1,t_2,...,t_{2m+2}) = 1.$$

This is rather an easy optimisation problem with algebraic constraints. Once one knows the number of intervals *m*, this optimization problem can be solved by using standard numerical techniques such as gradient methods, simplex-based search methods, genetic algorithms, etc. In simple cases, the solution can be obtained in an analytical form as it takes place in the example below.

The number of intervals in which the failure rate remains constant is a priori unknown. In the following we suggest an algorithm, similar to that introduced in [12] and [16], which solves this problem. We start with the verification if only one of the two $\overline{\lambda}$ or $\underline{\lambda}$ for the whole time period [0, *T*] satisfies the constraints. If the result is positive we can compute the value of the objective function. Then we set *m* = 0, solve the optimization problem and compare the obtained value of the objective function with the previous result. If it is different, we may continue and increase *m* by 1, and so on. The process will be stopped if the expression for the density function $\rho(t)$ does not change (or changes negligibly) and the improvement of the objective function also is not observed.

5 Example

Assume we are interested in knowing bounds $\underline{\tau}$ and $\overline{\tau}$ on the mean time to failure $\tau = \int_0^\infty t\rho(t)dt$ of a system and the following data (constraints) are known:

$$\Pr(q) = 1 - \int_0^\infty I_{[0,q]}(t)\rho(t)dt = p \text{ and } \underline{\lambda} \le \lambda(t) \le \overline{\lambda}.$$

That is, we know precisely the probability $\Pr(q)$, which we interpret as system's reliability at time *q*, and the lower $\underline{\lambda}$ and upper bound $\overline{\lambda}$ on the failure rate. $I_{[0,q]}(t)$ is the indicator function equal to 1 if $t \in [0,q]$ or equal to 0 otherwise. The consistency relation between the reliability and failure rate is expressed by the two inequalities $\exp(-\overline{\lambda}q) \le p \le \exp(-\underline{\lambda}q)$. If $\exp(-\overline{\lambda}q) = p$ or $p = \exp(-\underline{\lambda}q)$, the solution to the problem is simple, as there is only one pdf satisfying the either equality. The problem of this kind was described in [17]. This problem becomes more complicated if the strong inequalities hold $\exp(-\overline{\lambda}q) < p < \exp(-\underline{\lambda}q)$. For

this case, there are intervals on which the failure rate switches. Hence we start with $m = 0$. However, immediately it becomes clear that for $m = 0$ the expression for $\rho(t)$ contains only one unknown parameter t_1 while there are two constraints

$$\Pr(q) = 1 - \int_0^\infty I_{[0,q]}(t)\rho(t)dt = p\,,\ \int_0^\infty \rho(t)dt = 1.$$

This is why we have to increase m by 1

Determining $\underline{\tau}$. The graph of the pdf, $\rho(t)$, for which τ attains its minimum takes the form as shown in Fig. 2:

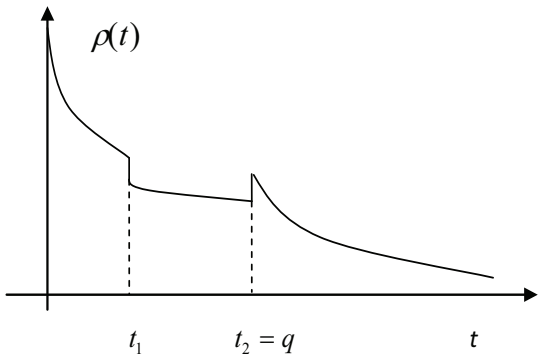

Figure 2. The behaviour of the pdf for which τ attains its minimum

Hence $\Pr(q) = \exp(-\overline{\lambda}t_1)\cdot\exp\left(-\underline{\lambda}(q - t_1\right) = p$.

From this equation we obtain

$$t_1 = -\frac{1}{\overline{\lambda} - \underline{\lambda}}\ln\left(p\cdot\exp(\underline{\lambda}q)\right) = -\frac{1}{\overline{\lambda} - \underline{\lambda}}\left(\ln p + \underline{\lambda}q\right)$$

Now we compute the value of t_2. First, assume that $t_2 \neq q$ (e.g. $t_2 > q$). Then the following equation must hold:

$$1 - p + \underline{\lambda}p\int_q^{t_2}\left(\exp\left(-\underline{\lambda}(t-q)\right)\right)dt +$$

$$\overline{\lambda}p(1 - \exp\left(-\underline{\lambda}(t_2 - q)\right)\int_{t_2}^\infty\left(\exp\left(-\overline{\lambda}(t - t_2)\right)\right)dt = 1.$$

It is true if $t_2 = q$. Finally,

$$\underline{\tau} = \int_0^\infty P(t)dt = \int_0^{t_1}\exp(-\overline{\lambda}t)dt +$$

$$\exp(-\overline{\lambda}t_1)\cdot\int_{t_1}^q\exp\left(-\underline{\lambda}(t - t_1)\right)dt + p\cdot\int_q^\infty\exp\left(-\overline{\lambda}(t - q)\right)dt =$$

$$\frac{1}{\overline{\lambda}}\left(1 + p - \exp(-\overline{\lambda}t_1)\right) + \frac{1}{\underline{\lambda}}\left(1 - \exp(-\underline{\lambda}(q - t_1)\right)\cdot\exp(-\overline{\lambda}t_1).$$

Increasing m by 1 does not lead to any improvement. Thus the obtained formula value is optimal one.

Determining $\overline{\tau}$. The graph of the pdf, $\rho(t)$, for which τ attains its maximum takes the form as shown in Fig. 3.

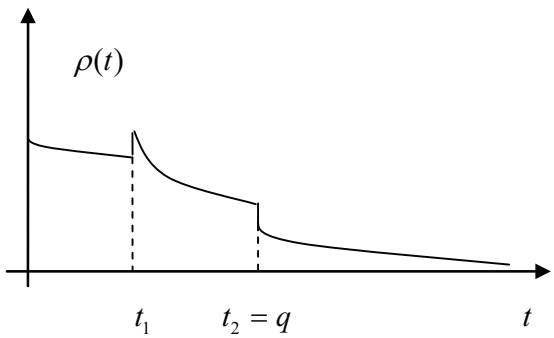

Figure 3. The behaviour of the pdf for which τ attains its maximum

For this case we can perform computations similar to the above and arrive at the result

$$\overline{\tau} = \frac{1}{\underline{\lambda}}\left(1 + p - \exp(-\underline{\lambda}t_1)\right) + \frac{1}{\overline{\lambda}}\left(1 - \exp(-\overline{\lambda}(q - t_1)\right)\cdot\exp(-\underline{\lambda}t_1).$$

6 Concluding notes

In spite of the existence of a number of risk/reliability and other applied models built on imprecise statistical reasoning, only a few of them have ever been used in practice – and then only hesitantly –, the rest remaining firmly in the academic realm. Perhaps the complexity of imprecise statistical reasoning as a whole is such as to severely limit the accessibility of this kind of models to potential practitioners. We nevertheless believe that the main obstacles to the practical application of this knowledge are different. One which is thoroughly familiar to the group of experts who practise interval computations and which we have repeatedly mentioned [12], [16]: it is namely the rapid growth in imprecision that occurs when intervals are propagated through mathematical models and when the number of components in a system is large. The other one stems

from the requirement of imprecise probability theory that the random value is to be bounded. This requirement appears very restrictive for reliability applications, as some reliability models explicitly contain an upper bound on time to failure which is in reality an arbitrary value.

Our main finding was that bounding the failure rate allows deriving reliability measures devoid of an upper bound on time to failure. That is, the sample space of time to failure is now as it must be from zero to infinity. This is the basic assumption on which all conventional reliability models rest and deviations from that can hardly be practical. Making judgements on the lower and upper bounds of failure rates is meaningful and can often be substantiated by observed events taking place in the system of interest or analogous ones.

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Appendix

Theorem. *If for any interval $\alpha \le t \le \beta$, $0 \le \alpha < \beta \le T$ and for any $h_0, h_1, ..., h_n \in \mathbf{R}$ it holds that*

$$g(t) \ne h_0 + \sum_{i=1}^{n} h_i f_i(t), \tag{8}$$

then the failure rate $\lambda(t)$, on which inf and sup are attained in problems (3) subject to constraints (4), (5) and (7), is a step-wise function which is equal either to $\underline{\lambda}$ or to $\overline{\lambda}$.

Proof. According to the method of Lagrange [18] the primal form of optimization problem (3) subject to constraints (4), (5) and (7) is to be replaced by the equivalent unconstrained optimization problem. To do so the following function is introduced

$$J^*(t) = g(t)y'(t)\exp(-y(t)) + \sum_{i=1}^{n} \mu_i f_i(t) y'(t)\exp(-y(t)) + \mu_0 y'(t)\exp(-y(t)) + \mu^*(t)\left(y'(t) - u^2(t)\right) + \mu^{**}(t)\left(y'(t) + v^2(t)\right)$$

Where $\mu, i \le n$ and $\mu^*(t)$, $\mu^{**}(t)$ are unknown Lagrange multipliers.

Then the Euler-Lagrange equations (the necessary condition of optimality) take the form:

$$\frac{\partial J^*}{\partial y} - \frac{d}{dt}\left(\frac{\partial J^*}{\partial y'}\right) = 0; \qquad \frac{\partial J^*}{\partial u} = 0; \qquad \frac{\partial J^*}{\partial v} = 0.$$

In our case these equations become:

$$-g(t)y'(t)\exp(-y(t)) - \sum_{i=1}^{n} \mu_i f_i(t) y'(t)\exp(-y(t)) - \mu_0 y'(t)\exp(-y(t)) + g(t)y'(t)\exp(-y(t)) + \sum_{i=1}^{n} \mu_i f_i(t) y'(t)\exp(-y(t)) + \mu_0 y'(t)\exp(-y(t)) - g'(t)\exp(-y(t)) - \sum_{i=1}^{n} \mu_i f_i'(t)\exp(-y(t)) + d\left(\mu^*(t)\right)/dt + d\left(\mu^{**}(t)\right)/dt = 0$$

$\mu^*(t)u(t) = 0$ and $\mu^{**}(t)v(t) = 0$

Here

$$g'(t) = dg(t)/dt;\ f_i'(t) = df_i(t)/dt, i \le n;$$
$$y''(t) = d^2 y(t)/dt^2.$$

It can be concluded that if $u(t) \ne 0$ and $v(t) \ne 0$ simultaneously then $\mu^*(t) = \mu^{**}(t) = 0$. Hence $d\left(\mu^*(t)\right)/dt = 0$ and $d\left(\mu^{**}(t)\right)/dt = 0$ resulting in

$$g'(t) + \sum_{i=1}^{n} \mu_i f_i'(t) = 0,$$

or after integration

$$g(t) + \sum_{i=1}^{n} \mu_i f_i(t) + c = 0, \tag{9}$$

in which c is arbitrary constant. (9) contradicts to (8). To resolve this conflict, one of the functions $u(t)$, $v(t)$ must be equal to zero inside the interval $\alpha \le t \le \beta$. On the other hand, they cannot be both equal to zero because the equalities $\lambda(t) = \underline{\lambda}$ and $\lambda(t) = \overline{\lambda}$ cannot hold simultaneously.

Finally, we conclude that the failure rate alternates between $\underline{\lambda}$ and $\overline{\lambda}$ within the time period $[0,T]$.