Modelling uncertainties in limit state functions

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Abstract

In this paper uncertainties in limit state functions g as arising in engineering problems are modelled by adding additional parameters and by introducing parameterized probability density functions which describe the uncertainties of these new additional parameters and of the basic variables of g. This will lead to a function $p_f(a,b)$ for the probability of failure depending on parameters a and b corresponding to the two parameterized density functions. Further the parameters a and b are assumed to be uncertain. Using intervals, sets or random sets to model their uncertainty results in upper probabilities $\overline{p}_f$ of failure. In this context we also discuss different notions of independence such as strong independence, epistemic irrelevance and random set independence and present a simple engineering example.

Keywords. Probability of failure, limit state functions, parameterized probability measures, random sets, random set independence, epistemic irrelevance, strong independence.

1 Introduction

In reliability analysis the probability p_f of failure of a system is obtained by

$$p_f = \int\limits_{\{x\colon g(x)\leq 0\}} f^X(x)\,\mathrm{d}x \tag{1}$$

where $x = (x_1,\ldots,x_n)$ are the basic variables of the system such as material properties and loads and where f^X is a probability density function describing the uncertainty of the variables x. The function g is the limit state function of the system telling us for which x the system fails ($g(x) \leq 0$) or not ($g(x) > 0$), see also [14].

In the case of scarce information about the values of the basic variables x and the behaviour of the system it may be neither sufficient to model the uncertainty of the variables x by a single probability density f^X nor to describe the system's reliability by a single deterministic limit state function g. To overcome such difficulties, fuzzy sets [17], random sets [3], credal sets [13] or sets of parameterized probability measures [9] have been used to model the uncertainty of the variables x, cf. also [6, 8, 10, 11]. Uncertainties in the limit state function g have been modelled using additional random variables [5], fuzzy sets, random sets [12] or fuzzy probabilities [1, 15].

The aim of this paper is to develop a function

$$p_f(a,b) = \iint\limits_{\{(x,z)\colon h(x,z)\leq 0\}} f_b^Z(z)\,\mathrm{d}z\, f_a^X(x)\,\mathrm{d}x \tag{2}$$

depending on vectors of parameters a and b parameterizing the probability density functions f_a^X and f_b^Z. These density functions describe the uncertainty of the basic variables x and the additional parameters z of an extended limit state function h. These additional variables z are used to parameterize a familiy of limit state functions g_z with $g_z(x) = h(x,z)$.

In a next step we assume that the parameters a and b are uncertain themselves modelling their uncertainty by intervals, sets or random sets. This approach gives us the possibility to describe the uncertainty of x and z by sets of probability measures generated by the density functions f_a^X and f_b^Z and their uncertain parameters a and b. The functions f_a^X and f_b^Z allow us to use more specific probability measures such as Gaussian distributions in contrast to the case where the uncertainty of x and z is directly modelled by sets or random sets. Such coarser models of uncertainty are also encompassed simply by replacing f_a^X and f_b^Z by Dirac measures.

A simple engineering example with one uncertain basic variable x exemplifies different cases and models of uncertainty of a and b and the computation of the upper probability $\overline{p}_f$ of failure by means of $p_f(a,b)$ with respect to different notions of indepence between the limit state functions and the basic variables.

2 Uncertain limit state functions

2.1 Limit state functions

In reliability theory a system and its corresponding continuous *limit state function*

$$g : \mathcal{X} \subseteq \mathbb{R}^n \to \mathcal{Y} \subseteq \mathbb{R} : \; x \to y = g(x) \tag{3}$$

is given with output $y \in \mathcal{Y}$ depending on a vector of *n basic variables* $x = (x_1, \ldots, x_n) \in \mathcal{X} \subseteq \mathbb{R}^n$ where $g(x) \leq 0$ means failure of the system. The *probability p_f of failure* of the system is then defined by

$$p_f = P(g(X) \leq 0) = \int_{\mathcal{X}} \chi(g(x) \leq 0) f^X(x) \,\mathrm{d}x \tag{4}$$

where f^X is the joint probability density function of the basic random variables $X = (X_1, \ldots, X_n)$ and where

$$\chi(\text{expression}) = \begin{cases} 1 & \text{expression true,} \\ 0 & \text{expression false.} \end{cases} \tag{5}$$

The set $R_f = \{x \in \mathcal{X} : \; g(x) \leq 0\}$ is the *failure region* of the system which we describe by the indicator function

$$q : \mathcal{X} \to \{0,1\} : x \to \chi(g(x) \leq 0). \tag{6}$$

2.2 Parameterized limit state functions

We parameterize the limit state function $g : \mathcal{X} \to \mathcal{Y}$ by means of a vector $z = (z_1, \ldots, z_m) \in \mathcal{Z} \subseteq \mathbb{R}^m$ of additional parameters using a function

$$h : \mathcal{X} \times \mathcal{Z} \to \mathcal{Y} : (x,z) \to h(x,z) \tag{7}$$

where again $h(x,z) \leq 0$ means failure. A function

$$g_z : \mathcal{X} \to \mathcal{Y} : x \to g_z(x) = h(x,z) \tag{8}$$

is then one of the available limit state functions specified by a parameter value z. When both the basic variables x and the parameters z are uncertain, the probability p_f of failure is defined by

$$p_f = \int_{\mathcal{X}} \int_{\mathcal{Z}} \chi(h(x,z) \leq 0) f^{X,Z}(x,z) \,\mathrm{d}z \,\mathrm{d}x \tag{9}$$

where $f^{X,Z} : \mathcal{X} \times \mathcal{Z} \to \mathbb{R}$ is the joint density function of the random variables $X = (X_1, \ldots, X_n)$ and $Z = (Z_1, \ldots, Z_m)$. The uncertainty of the parameters z is the uncertainty in the choice of an appropriate limit state function g_z.

2.3 Independence of the basic variables and the parameters

In the following we always assume that the random variables X and Z are *independent* which has the following meaning:

(a) *If we learn the values of the basic variables x, our knowledge about the parameters z and therefore about the choice of the limit state functions g_z does not change.*

(b) *Learning the values of the parameters z and therefore learning which limit state function g_z to use has no influence on our knowledge about the variables x.*

Then the probability p_f of failure is given by

$$p_f = \int_{\mathcal{X}} \int_{\mathcal{Z}} \chi(h(x,z) \leq 0) f^Z(z) \,\mathrm{d}z \, f^X(x) \,\mathrm{d}x \tag{10}$$

with density functions f^X and f^Z for their corresponding random variables X and Z. The inner integral is a function

$$q : \mathcal{X} \to [0,1] : x \to \int_{\mathcal{Z}} \chi(h(x,z) \leq 0) f^Z(z) \,\mathrm{d}z \tag{11}$$

which is a generalization of q in Eq. (6). For q in Eq. (6) only the function values 1 and 0 are admissible telling us wether an $x \in \mathcal{X}$ is in the failure region R_f or not, but here q describes an uncertain failure region similar to a membership function of a fuzzy set. The value $q(x)$ is the probability that x belongs to the failure region R_f.

2.4 Sets of probability measures and notions of independence

We use now sets $\mathcal{M}_X$ and $\mathcal{M}_Z$ of probability measures to describe the uncertainty of the basic variables x and parameters z of the limit state function h. Since we want to keep the assumption that the basic variables x and the limit state functions g_z are independent we have to compute the upper probability of failure with respect to the different notions of independence for sets of probability measures [2, 6]. We consider here *strong independence*, the weaker and asymmetric *epistemic irrelevance* and later on in Sec. 3.3 *random set independence*.

Strong independence [2, 6, 16]: It is the most restrictive definition of independence simply considering all possible product measures $P_X \otimes P_Z$ for $P_X \in \mathcal{M}_X$ and $P_Z \in \mathcal{M}_Z$. Then the upper probability $\overline{p}_f^{\mathrm{S}}$ of failure in case of strong independence is obtained by

$$\begin{aligned} \overline{p}_f^{\mathrm{S}} &= \sup\big\{(P_X \otimes P_Z)(S_f) : \; P_X \in \mathcal{M}_X, P_Z \in \mathcal{M}_Z\big\} \\ &= \sup_{\substack{P_X \in \mathcal{M}_X \\ P_Z \in \mathcal{M}_Z}} \int_{\mathcal{X}} \int_{\mathcal{Z}} \chi(h(x,z) \leq 0) \,\mathrm{d}P_Z(z) \,\mathrm{d}P_X(x) \\ &= \sup_{\substack{P_X \in \mathcal{M}_X \\ q \in Q}} \int_{\mathcal{X}} q(x) \,\mathrm{d}P_X(x) \end{aligned} \tag{12}$$

where $S_f = \{(x,z) : h(x,z) \leq 0\}$ and Q the set

$$Q = \Big\{ q : \mathcal{X} \to [0,1] : \\ q(x) = \int_{\mathcal{Z}} \chi(h(x,z) \leq 0)\, \mathrm{d}P_Z(z),\, P_Z \in \mathcal{M}_Z \Big\} \quad (13)$$

of all functions q describing the uncertainty of the failure region R_f as in Eq. (11).

Epistemic irrelevance [2, 4, 16]: We start with the above formula for $\overline{p}_f^{\mathrm{S}}$, but move then $\sup_{P_Z \in \mathcal{M}_Z}$ inside the outer integral:

$$\begin{aligned} \overline{p}_f^{\mathrm{S}} &= \sup_{\substack{P_X \in \mathcal{M}_X \\ P_Z \in \mathcal{M}_Z}} \int_{\mathcal{X}} \int_{\mathcal{Z}} \chi(h(x,z) \leq 0)\, \mathrm{d}P_Z(z)\, \mathrm{d}P_X(x) \\ &\leq \sup_{P_X \in \mathcal{M}_X} \int_{\mathcal{X}} \sup_{P_Z \in \mathcal{M}_Z} \int_{\mathcal{Z}} \chi(h(x,z) \leq 0)\, \mathrm{d}P_Z(z)\, \mathrm{d}P_X(x) \\ &= \sup_{P_X \in \mathcal{M}_X} \int_{\mathcal{X}} \overline{q}(x)\, \mathrm{d}P_X(x) =: \overline{p}_f^{X \not\to Z} \end{aligned} \quad (14)$$

with

$$\overline{q}(x) = \sup_{P_Z \in \mathcal{M}_Z} \int_{\mathcal{Z}} \chi(h(x,z) \leq 0)\, \mathrm{d}P_Z(z) = \sup_{q \in Q} q(x). \quad (15)$$

The result is a formula for the upper probability $\overline{p}_f^{X \not\to Z}$ in case of epistemic irrelevance, because for each x we can choose its own $P_Z \in \mathcal{M}_Z$ or more exactly a conditional probability measure $P_Z(\,\cdot\,|x)$ given x. The notation $X \not\to Z$ means that X is epistemically irrelevant to Z, see [4], or in our case that the basic variables are epistemically irrelevant to the parameterized limit state functions g_z. Epistemic irrelevance is an asymmetric notion of independence meaning only what we have stated in (a) in Sec. 2.3, but not necessarily the other way round as in (b). The set $\mathcal{M}_{X \not\to Z}$ of all probability measures according to epistemic irrelevance of X to Z is defined by

$$\mathcal{M}_{X \not\to Z} = \Big\{ P : P(E) = \int_{\mathcal{X}} \int_{\mathcal{Z}} \chi((x,z) \in E)\, \mathrm{d}P_Z(z|x)\, \mathrm{d}P_X(x), \\ P_X \in \mathcal{M}_X,\, P_Z(\,\cdot\,|x) \in \mathcal{M}_Z \Big\} \quad (16)$$

where E is an event. In Eq. (14) we write $P_Z(z)$ instead of $P_Z(z|x)$ since it is clear that we use different probability measures P_Z and not only one because of the $\sup_{P_Z \in \mathcal{M}_Z}$ in the formula.

When it is possible to assume epistemic irrelevance we have the advantage that we can treat the uncertainty of the basic variables and of the limit state functions completely separately. We can compute $\overline{q}$ in advance and then using $\overline{q}$ for different models of uncertainty of x.

The function $\overline{q}$ is the upper envelope of the set Q defined in Eq. (13). If this upper envelope $\overline{q}$ is an element of Q we have $\overline{p}_f^{\mathrm{S}} = \overline{p}_f^{X \not\to Z}$.

3 The probability of failure $p_f(a,b)$ with uncertain parameters a and b

3.1 The function $p_f(a,b)$

Let us now extend Equation (10) by adding parameters $a = (a_1, \ldots, a_{n_a}) \in \mathbb{R}^{n_a}$ for the probability density function f^X describing the uncertainty of the basic variables x and parameters $b = (b_1, \ldots, b_{n_b}) \in \mathbb{R}^{n_b}$ for the density f^Z of the additional parameters z which leads to a function

$$p_f(a,b) = \int_{\mathcal{X}} \int_{\mathcal{Z}} \chi(h(x,z) \leq 0) f_b^Z(z)\, \mathrm{d}z\, f_a^X(x)\, \mathrm{d}x. \quad (17)$$

This function $p_f(a,b)$ provides an interface for controlling the shape of the probability density functions used for modelling the uncertainty of the basic variables x and the parameters z. We also write $p_f(a,b; f_a^X, f_b^Z)$ if it is necessary to emphasize which density functions are used.

In the following the parameters a and b are assumed to be uncertain; intervals, sets or random sets are used to describe their uncertainty. This and the approach with parameterized density functions f_a^X and f_b^Z give us a convenient way to generate the sets $\mathcal{M}_X$ and $\mathcal{M}_Z$ of probability measures and the possibility to model the uncertainty of x and z by means of more specific probability measures than directly using sets or random sets for x and z. An example for such a parameterized density f_a^X or f_b^Z is the density of a Gaussian distribution depending on expectation μ and variance σ^2. Then describing the uncertainty of μ and σ by sets or random sets leads to sets $\mathcal{M}_X$ or $\mathcal{M}_Z$ of probability measures.

3.2 Uncertainty of the parameters a and b modelled by sets A and B

We describe the uncertainty of the parameter $a \in \mathbb{R}^{n_a}$ by a set $A \subseteq \mathbb{R}^{n_a}$ and the uncertainty of $b \in \mathbb{R}^{n_b}$ by a set $B \subseteq \mathbb{R}^{n_b}$ and show how the upper probability of failure is determined in case of strong independence or epistemic irrelevance. But first we have to generate the sets of probability measures $\mathcal{M}_X$ and $\mathcal{M}_Z$.

Generating $\mathcal{M}_X$ *and* $\mathcal{M}_Z$:

$$\mathcal{M}_X = \Big\{ P : P(E) = \int_{\mathcal{A}} \int_{\mathcal{X}} \chi(x \in E) f_a^X(x)\, \mathrm{d}x\, \mathrm{d}P_A(a), \\ P_A \in \mathcal{M}(A) \Big\} \quad (18)$$

where $\mathcal{M}(A) = \{P : P(A) = 1\}$ is the set of all probability measures living on the set A and where E is an event. The set $\mathcal{M}_Z$ is generated in an analogous way using f_b^Z and $\mathcal{M}(B)$.

Strong independence: Eq. (12) together with Eq. (18) leads to the following formula for the upper probability

$\overline{p}_f^{\mathrm{S}}$ in case of strong independence:

$$\begin{aligned}\overline{p}_f^{\mathrm{S}} &= \sup_{\substack{P_X\in\mathcal{M}_X\\ P_Z\in\mathcal{M}_Z}} \int_{\mathcal{X}}\int_{\mathcal{Z}} \chi(h(x,z)\le 0)\,\mathrm{d}P_Z(z)\,\mathrm{d}P_X(x)\\
&= \sup_{\substack{P_A\in\mathcal{M}(A)\\ P_B\in\mathcal{M}(B)}} \int_{\mathcal{A}}\int_{\mathcal{X}}\int_{\mathcal{B}}\int_{\mathcal{Z}} \chi(h(x,z)\le 0) f_b^Z(z)\,\mathrm{d}z\cdot\\
&\qquad\cdot\,\mathrm{d}P_B(b) f_a^X(x)\,\mathrm{d}x\,\mathrm{d}P_A(a)\\
&= \sup_{\substack{P_A\in\mathcal{M}(A)\\ P_B\in\mathcal{M}(B)}} \int_{\mathcal{A}}\int_{\mathcal{B}} p_f(a,b)\,\mathrm{d}P_B(b)\,\mathrm{d}P_A(a) \qquad (19)\\
&= \sup_{\substack{a\in A\\ b\in B}} \int_{\mathcal{A}}\int_{\mathcal{B}} p_f(\xi,\eta)\,\mathrm{d}\delta_b(\eta)\,\mathrm{d}\delta_a(\xi) = \sup_{\substack{a\in A\\ b\in B}} p_f(a,b).\end{aligned}$$

The Dirac measures δ_a and δ_b are extreme points in the sets $\mathcal{M}(A)$ and $\mathcal{M}(B)$ of probability measures.

Epistemic irrelevance: Eq. (14) together with Eq. (18) leads to the formula for the upper probability $\overline{p}_f^{X\not\to Z}$ in case of epistemic irrelevance:

$$\begin{aligned}\overline{p}_f^{X\not\to Z} &= \sup_{P_A\in\mathcal{M}(A)} \int_{\mathcal{A}}\int_{\mathcal{X}} \sup_{P_B\in\mathcal{M}(B)} \int_{\mathcal{B}}\int_{\mathcal{Z}} \chi(h(x,z)\le 0) f_b^Z(z)\cdot\\
&\qquad\cdot\,\mathrm{d}z\,\mathrm{d}P_B(b) f_a^X(x)\,\mathrm{d}x\,\mathrm{d}P_A(a)\\
&= \sup_{a\in A}\int_{\mathcal{X}} \sup_{b\in B}\int_{\mathcal{Z}} \chi(h(x,z)\le 0) f_b^Z(z)\,\mathrm{d}z\, f_a^X(x)\,\mathrm{d}x\\
&= \sup_{a\in A}\int_{\mathcal{X}} \overline{q}(x) f_a^X(x)\,\mathrm{d}x \qquad (20)\end{aligned}$$

with $\overline{q}(x) = \sup_{b\in B}\int_{\mathcal{Z}}\chi(h(x,z)\le 0) f_b^Z(z)\,\mathrm{d}z$, again using that δ_a and δ_b are extreme points in $\mathcal{M}(A)$ and $\mathcal{M}(B)$.

3.3 Uncertainty of *a* and *b* modelled by random sets $\mathscr{A}$ and $\mathscr{B}$

A *random set* as introduced by [3] is a family $\mathscr{A}$ of *focal sets* A_i together with *weights* $m_{\mathscr{A}}(A_i)$ which sum up to one. Then the upper probability $\overline{P}(E)$ or *plausibility* $\mathrm{Pl}_{\mathscr{A}}(E)$ of an event E is given in the case of a finite random set with focal sets $A_1,\ldots,A_{|\mathscr{A}|}$ by the formula

$$\overline{P}(E) = \mathrm{Pl}_{\mathscr{A}}(E) = \sum_{i:\,E\cap A_i\neq\varnothing} m_{\mathscr{A}}(A_i) = \sum_{i=1}^{|\mathscr{A}|} m_{\mathscr{A}}(A_i) \sup_{P\in\mathcal{M}(A_i)} P(E) \qquad (21)$$

where $|\mathscr{A}|$ denotes the number of focal sets where

$$\mathcal{M}(A_i) = \{P:\ P(A_i)=1\} \qquad (22)$$

is the set of all probability measures on the focal set A_i, cf. [6]. The lower probability $\underline{P}(E)$ or belief $\mathrm{Bel}_{\mathscr{A}}(E)$ is defined by

$$\underline{P}(E) = \mathrm{Bel}_{\mathscr{A}}(E) = \sum_{i:\,A_i\subseteq E} m_{\mathscr{A}}(A_i) = \sum_{i=1}^{|\mathscr{A}|} m_{\mathscr{A}}(A_i) \inf_{P\in\mathcal{M}(A_i)} P(E). \qquad (23)$$

First we have to generate the sets $\mathcal{M}_X$ and $\mathcal{M}_Z$ by means of random sets $\mathscr{A}$ and $\mathscr{B}$ modelling the uncertainty of a and b. Then we show how to determine the upper probabilities of failure for strong independence, epistemic irrelevance and random set independence.

Generating the sets $\mathcal{M}_X$ and $\mathcal{M}_Z$:

$$\begin{aligned}\mathcal{M}_X &= \Big\{P:\ P(E) = \int_{\mathcal{A}}\int_{\mathcal{X}} \chi(x\in E) f_a^X(x)\,\mathrm{d}x\,\mathrm{d}P_A(a),\\
&\qquad P_A\in\mathcal{M}(\mathscr{A})\Big\}\\
&= \Big\{P:\ P(E) = \sum_{i=1}^{|\mathscr{A}|} m_{\mathscr{A}}(A_i)\cdot \qquad (24)\\
&\qquad\cdot\int_{\mathcal{A}}\int_{\mathcal{X}} \chi(x\in E) f_a^X(x)\,\mathrm{d}x\,\mathrm{d}P_{A_i}(a),\\
&\qquad P_{A_i}\in\mathcal{M}(A_i), i=1,\ldots,n\Big\}\end{aligned}$$

where $\mathcal{M}(\mathscr{A})$ is the set of all probability measures generated by a random set $\mathscr{A}$, cf. [9]. A probability measure in $\mathcal{M}(\mathscr{A})$ is a weighted sum of probability measures $P_{A_i}\in\mathcal{M}(A_i)$ living on the focal sets A_i. The set $\mathcal{M}_Z$ is obtained in a similar way using f_b^Z and the random set $\mathscr{B}$.

Strong independence: Eq. (12) together with Eq. (24) leads to the upper probability

$$\begin{aligned}\overline{p}_f^{\mathrm{S}} &= \sup_{\substack{P_{A_r}\in\mathcal{M}(A_r),r=1,\ldots,|\mathscr{A}|\\ P_{B_s}\in\mathcal{M}(B_s),s=1,\ldots,|\mathscr{B}|}} \sum_{i=1}^{|\mathscr{A}|}\sum_{j=1}^{|\mathscr{B}|} m_{\mathscr{A}}(A_i) m_{\mathscr{B}}(B_j)\cdot \qquad (25)\\
&\qquad\cdot\int_{\mathcal{A}}\int_{\mathcal{B}} p_f(a,b)\,\mathrm{d}P_{A_i}(a)\,\mathrm{d}P_{B_j}(b)\\
&= \sup_{\substack{a_r\in A_r,r=1,\ldots,|\mathscr{A}|\\ b_s\in B_s,s=1,\ldots,|\mathscr{B}|}} \sum_{i=1}^{|\mathscr{A}|}\sum_{j=1}^{|\mathscr{B}|} m_{\mathscr{A}}(A_i) m_{\mathscr{B}}(B_j) p_f(a_i,b_j)\end{aligned}$$

in case of strong independence replacing the probability measures P_{A_i} and P_{B_j} by Dirac measures δ_{a_i} and δ_{b_j} on their corresponding focal sets A_i and B_j similar to the section before. A general proof that the upper probability can be obtained by means of Dirac measures can be found in [7].

Epistemic irrelevance: Eq. (14) together with Eq. (24) results in the upper probability $\overline{p}_f^{X\not\to Z}$ in case of epistemic irrelevance:

$$\begin{aligned}\overline{p}_f^{X\not\to Z} &= \sum_{i=1}^{|\mathscr{A}|} m_{\mathscr{A}}(A_i) \sup_{P_A\in\mathcal{M}(A_i)} \int_{\mathcal{A}}\int_{\mathcal{X}} \sum_{j=1}^{|\mathscr{B}|} m_{\mathscr{B}}(B_j)\cdot \qquad (26)\\
&\qquad\cdot\sup_{P_B\in\mathcal{M}(B_j)} \int_{\mathcal{B}}\int_{\mathcal{Z}} \chi(h(x,z)\le 0) f_b^Z(z)\cdot\\
&\qquad\cdot\,\mathrm{d}z\,\mathrm{d}P_B(b)\, f_a^X(x)\,\mathrm{d}x\,\mathrm{d}P_A(a)\\
&= \sum_{i=1}^{|\mathscr{A}|} m_{\mathscr{A}}(A_i) \sup_{a\in A_i}\int_{\mathcal{X}} \sum_{j=1}^{|\mathscr{B}|} m_{\mathscr{B}}(B_j)\cdot\\
&\qquad\cdot\sup_{b\in B_j}\int_{\mathcal{Z}} \chi(h(x,z)\le 0) f_b^Z(z)\,\mathrm{d}z\, f_a^X(x)\,\mathrm{d}x.\end{aligned}$$

The function $\overline{q}$ is given here by

$$\overline{q}(x) = \sum_{j=1}^{|\mathscr{B}|} m_{\mathscr{B}}(B_j) \sup_{b \in B_j} \int_{\mathcal{Z}} \chi(h(x,z) \le 0) f_b^Z(z)\,\mathrm{d}z. \tag{27}$$

Random set independence: Let the uncertainty of a variable a be modelled by a random set $\mathscr{A}$ with focal sets A_i and weights $m_{\mathscr{A}}(A_i)$ and the uncertainty of a variable b by a random set $\mathscr{B}$ with focal sets B_j and weights $m_{\mathscr{B}}(B_j)$. The *joint random set*, in the classical version assuming *random set independence*, is defined as the family $\mathscr{C}$ of all Cartesian products $C_{ij} = A_i \times B_j$ of focal sets A_i and B_j. The weights of these *joint focal sets* C_{ij} are given by the product $m_{\mathscr{C}}(C_{ij}) = m_{\mathscr{A}}(A_i) m_{\mathscr{B}}(B_j)$ and the formula for the *joint plausibility measure* Pl by

$$\begin{aligned} \mathrm{Pl}(E) &= \sum_{i=1}^{|\mathscr{A}|} \sum_{j=1}^{|\mathscr{B}|} m_{\mathscr{A}}(A_i) m_{\mathscr{B}}(B_j) \chi(E \cap (A_i \times B_j) \neq \varnothing) \\ &= \sum_{i=1}^{|\mathscr{A}|} \sum_{j=1}^{|\mathscr{B}|} m_{\mathscr{A}}(A_i) m_{\mathscr{B}}(B_j) \sup_{P \in \mathcal{M}(A_i \times B_j)} P(E) \end{aligned} \tag{28}$$

with set $\mathcal{M}(A_i \times B_j) = \{P : P(A_i \times B_j) = 1\}$, cf. [2, 3, 6]. This set is the largest possible set of joint probability measures generated by the marginal sets $\mathcal{M}(A_i)$ and $\mathcal{M}(B_j)$. The key properties of random set independence are that

(i) there are no interactions between focal sets A_i and B_j,
(ii) the focal sets A_i and B_j are chosen in a stochastically independent way;
(iii) the joint plausibility $\mathrm{Pl}(E)$ is obtained by solving optimization problems $\sup_{P \in \mathcal{M}(A_i \times B_j)} P(E)$ on each joint focal set $A_i \times B_j$ *separately* and *independently* of the other joint focal sets.

Our problem here is that density functions are involved in the formulas and that we have to combine not only two random sets $\mathscr{A}$ and $\mathscr{B}$ but also two density functions f_a^X and f_b^Z. So we have to generalize the formula for the joint plausibility measure. One possibility is to replace the set $\mathcal{M}(A_i \times B_j)$ by a set of joint probability measures generated by sets $\mathcal{M}_X^i$ and $\mathcal{M}_Z^j$ defined by

$$\begin{aligned} \mathcal{M}_X^i &= \Big\{P\colon P(E) = \int_{\mathcal{A}} \int_{E} f_a^X(x)\,\mathrm{d}x\,\mathrm{d}P_A(a),\, P_A \in \mathcal{M}(A_i)\Big\}, \\ \mathcal{M}_Z^j &= \Big\{P\colon P(E) = \int_{\mathcal{B}} \int_{E} f_b^Z(z)\,\mathrm{d}z\,\mathrm{d}P_B(b),\, P_B \in \mathcal{M}(B_j)\Big\} \end{aligned} \tag{29}$$

as the sets $\mathcal{M}_X$ and $\mathcal{M}_Z$ in Sec. 3.2. Now the question arises how to combine $\mathcal{M}_X^i$ and $\mathcal{M}_Z^j$. Since $\mathcal{M}(A_i \times B_j)$ is the largest possible set of joint probability measures generated by $\mathcal{M}(A_i)$ and $\mathcal{M}(B_j)$ an analogous approach would be to use here the set of all possible joint probability measures generated by $\mathcal{M}_X^i$ and $\mathcal{M}_Z^j$. But this means to consider also all possible joint density functions with marginals f_a^X and f_b^Z which is not very attractive because of the computational effort and because independence is not taken into account on the level of the density functions.

Another approach is to combine $\mathcal{M}_X^i$ and $\mathcal{M}_Z^j$ according to strong independence or epistemic irrelevance as in Sec. 3.2 which means to replace $\sup_{P \in \mathcal{M}(A_i \times B_j)} P(E)$ in Eq. (28) by the results of Eq. (19) or of Eq. (20):

For strong independence, locally for each pair of sets $\mathcal{M}_X^i$ and $\mathcal{M}_Z^j$, we get the upper probability

$$\overline{p}_f^{\mathrm{RS}} = \sum_{i=1}^{|\mathscr{A}|} \sum_{j=1}^{|\mathscr{B}|} m_{\mathscr{A}}(A_i) m_{\mathscr{B}}(B_j) \sup_{a \in A_i, b \in B_j} p_f(a,b) \tag{30}$$

by means of Eq. (19), cf. also [9]. We denote this upper probability by a superscript "RS" where "R" means random set independence and "S" indicates that the sets $\mathcal{M}_X^i$ and $\mathcal{M}_Z^j$ corresponding to $A_i \times B_j$ are combined according to strong independence. The difference to the "global" version of strong independence in Eq. (25) is that here the "sup" is inside instead of outside the sums. So it is clear that we have the ordering $\overline{p}_f^{\mathrm{S}} \le \overline{p}_f^{\mathrm{RS}}$.

Epistemic irrelevance, locally for each pair of sets $\mathcal{M}_X^i$ and $\mathcal{M}_Z^j$, leads to

$$\overline{p}_f^{\mathrm{R},X \not\to Z} = \sum_{i=1}^{|\mathscr{A}|} \sum_{j=1}^{|\mathscr{B}|} m_{\mathscr{A}}(A_i) m_{\mathscr{B}}(B_j) \sup_{a \in A_i} \int_{\mathcal{X}} \overline{q}_j(x) f_a^X(x)\,\mathrm{d}x \tag{31}$$

$$\overline{q}_j(x) = \sup_{b \in B_j} \int_{\mathcal{Z}} \chi(h(x,z) \le 0) f_b^Z(z)\,\mathrm{d}z, \tag{32}$$

cf. Eq. (20). We use here the superscript "$\mathrm{R},X \not\to Z$" instead of "RS" to denote the upper probability.

Summarizing the orderings of all upper probabilities we have $\overline{p}_f^{\mathrm{S}} \le \overline{p}_f^{X \not\to Z} \le \overline{p}_f^{\mathrm{R},X \not\to Z}$ and $\overline{p}_f^{\mathrm{S}} \le \overline{p}_f^{\mathrm{RS}} \le \overline{p}_f^{\mathrm{R},X \not\to Z}$.

We note that Dirac measures δ_a and δ_b instead of arbitrary density functions f_a^X and f_b^Z leads to the classical joint plausibility measure: For Dirac measures we always have $\mathcal{M}_X^i = \mathcal{M}(A_i)$, $\mathcal{M}_Z^j = \mathcal{M}(B_j)$. Further the resulting upper probabilities for $\mathcal{M}(A_i \times B_j)$ as in Eq. (28) or for sets of probability measures generated by $\mathcal{M}(A_i)$, $\mathcal{M}(B_j)$ according to strong independence or epistemic irrelevance coincides since Dirac measures $\delta_a \otimes \delta_b$, $a \in A_i$, $b \in B_j$ are extreme points in all these three sets. This means that we have $\overline{p}_f^{\mathrm{R}} := \overline{p}_f^{\mathrm{RS}} = \overline{p}_f^{\mathrm{R},X \not\to Z}$.

4 Alternative approaches and views

Let $Y_{|x} = h(x,Z)$ be the conditional random variable for the uncertain output of the parameterized limit state function h given a value of the basic variables x, Z the random variable corresponding to the parameters z, $f^{Y|x} : \mathcal{Y} \subseteq \mathbb{R} \to \mathbb{R}$ the probability density of $Y_{|x}$ and $F^{Y|x} : \mathcal{Y} \to [0,1]$ the probability distribution function. Then $q : \mathcal{X} \to [0,1]$ describing the uncertainty of the failure region is defined here by

$$q(x) = F^{Y|x}(0) = \int_{-\infty}^{0} f^{Y|x}(y)\,\mathrm{d}y \tag{33}$$

since $y \le 0$ means failure.

On the one hand the functions $f^{Y|x}$, $F^{Y|x}$ and q (or $\overline{F}^{Y|x}$ and $\overline{q}$ if sets of probability measures are used, see below) can be used to visualize the uncertainties in the limit state function. On the other hand the uncertainties in the limit state function can be specified providing these functions. Especially describing the uncertainty in the failure region by means of the function $\overline{q}$ in case of epistemic irrelevance opens the possibility to start also with fuzzy failure regions. Note that there may be a conceptual but not a formal difference between $\overline{q}$ and a membership function of a fuzzy set. To specify the limit state function g in its uncertain format instead of introducing additional parameters was also suggested in [12].

We show now how the two approaches are connected for the case that h is given by $y = h(x,z) = g(x) + z$ which means to add something uncertain to a deterministic limit state function g. Substituting $z = y - g(x)$ in Eq. (17) leads to

$$\begin{aligned} p_f(a,b) &= \int_{\mathcal{X}}\int_{\mathcal{Z}} \chi(g(x)+z \leq 0) f_b^Z(z)\,\mathrm{d}z\, f_a^X(x)\,\mathrm{d}x \\ &= \int_{\mathcal{X}}\int_{\mathcal{Y}} \chi(y \leq 0) f_b^Z(y-g(x))\,\mathrm{d}y\, f_a^X(x)\,\mathrm{d}x \\ &= \int_{\mathcal{X}}\int_{-\infty}^{0} f_{(g(x),b)}^{Y|x}(y)\,\mathrm{d}y\, f_a^X(x)\,\mathrm{d}x \\ &= \int_{\mathcal{X}} F_{(g(x),b)}^{Y|x}(0) f_a^X(x)\,\mathrm{d}x \end{aligned} \tag{34}$$

with $f_{(g(x),b)}^{Y|x}(y) = f_b^Z(y - g(x))$. The density $f_{(g(x),b)}^{Y|x}$ describes the uncertainty of the output of the limit state function and it is the same density function as f_b^Z, but moved from 0 to $g(x)$. This is indicated by the additional parameter $g(x)$ of the probability density $f_{(g(x),b)}^{Y|x}$ depending now on parameters which are not constant on $\mathcal{X}$. Modelling the uncertainty of parameter b by a set B we get an example for a function

$$\begin{aligned} \overline{q}(x) &= \sup_{b\in B} \int_{\mathcal{Z}} \chi(h(x,z) \leq 0) f_b^Z(z)\,\mathrm{d}z \\ &= \sup_{b\in B} F_{(g(x),b)}^{Y|x}(0) =: \overline{F}^{Y|x}(0) \end{aligned} \tag{35}$$

using both approaches. The function $\overline{F}^{Y|x}$ is the upper distribution function of $Y_{|x}$. In an analogous way we obtain the lower bound

$$\begin{aligned} \underline{q}(x) &= \inf_{b\in B} \int_{\mathcal{Z}} \chi(h(x,z) \leq 0) f_b^Z(z)\,\mathrm{d}z \\ &= \inf_{b\in B} F_{(g(x),b)}^{Y|x}(0) =: \underline{F}^{Y|x}(0). \end{aligned} \tag{36}$$

It is the lower probability of failure given $x \in \mathcal{X}$.

5 Numerical example

5.1 Problem statement

As a simple numerical example we consider a beam of length 3 m supported on both ends and additionally bedded on a spring, cf. Fig. 1. The values of the beam rigidity $EI = 1\,\mathrm{kNm}^2$ and of the load $f(\xi) = 100\,\mathrm{kN/m}$ are deterministic, but the value of the spring constant x (in our notation for the basic variables) is assumed to be uncertain.

The deterministic limit state function g is given as[1]

$$\begin{aligned} g(x) &= M_{\text{yield}} - \max_{\xi\in[0,3]} |M(\xi,x)| \\ &= M_{\text{yield}} - \frac{qL^2}{4} \max\left(\frac{(1-c(x))^2}{2}, c(x) - \frac{1}{2}\right) \end{aligned} \tag{37}$$

with $c(x) = 5x/(384EI/L^3 + 8x)$, see Fig. 1. $M(\xi,x)$ is the bending moment at $\xi \in [0,3]$ on the beam depending on the spring constant x and $M_{\text{yield}} = 21\,\mathrm{kNm}$ is the elastic limit moment of the beam for both positive and negative moments.

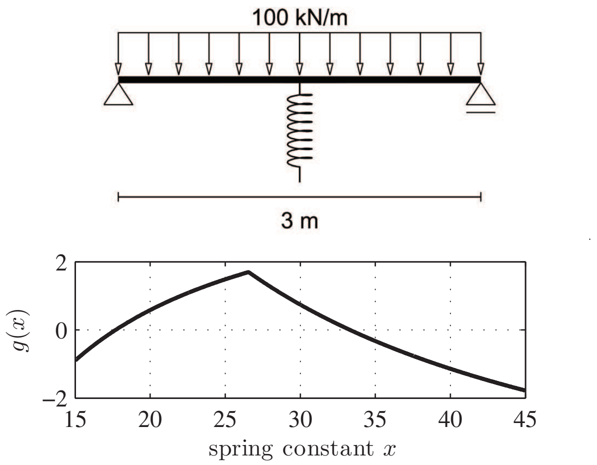

Figure 1: Beam bedded on a spring and deterministic limit state function g.

5.2 Modelling the uncertainty of spring constant x

The uncertainty of the spring constant x ([kN/m]) is modelled either by an interval A, by a random set $\mathscr{A}$ or by a Gaussian distribution. In the following we present what we will use for the basic variable x in the examples in the next section.

Interval A modelling the uncertainty of x:
We will use the interval $A = [\underline{a}, \overline{a}] = [20, 30]$ kN/m.

Random set $\mathscr{A}$ modelling the uncertainty of x:
The random set $\mathscr{A}$ is given by the focal sets $A_1 = [17, 30]$, $A_2 = [23, 31]$, $A_3 = [27, 32.5]$ and weights $m_{\mathscr{A}}(A_1) = 0.2$, $m_{\mathscr{A}}(A_2) = 0.3$ and $m_{\mathscr{A}}(A_3) = 0.5$.

[1] Thanks to one of the reviewers for providing an explicit formula.

Probability distribution modelling the uncertainty of the basic variable x:
We assume that x is Gaussian distributed with parameters $\mu = 34$ and $\sigma^2 = 1$.

6 Cases and examples

In this section we present examples and special cases with respect to the different notions of independence.

6.1 Sets of parameterized limit state functions

Let B be a set and

$$G = \{g_z : g_z(x) = h(x,z),\, z \in B\} \tag{38}$$

the family of limit state functions parameterized by $z \in B$. Further let the function $\underline{g}$ be the lower envelope of G defined by $\underline{g}(x) = \inf_{g_z \in G} g_z(x)$ and $\overline{g}$ the upper envelope.

In this case we have to set $f_b^Z := \delta_z$ and $b := z$ in Eqs. (19) and (20) which leads to

$$\begin{aligned}\overline{p}_f^{\mathrm{S}} &= \sup_{\substack{a\in A\\ z\in B}} p_f(x,z;f_a^X,\delta_z) \qquad (39)\\ &= \sup_{\substack{a\in A\\ z\in B}} \int_{\mathcal{X}}\int_{\mathcal{Z}} \chi(h(x,\eta)\le 0)\,\delta_z(\eta)\,\mathrm{d}\eta\, f_a^X(x)\,\mathrm{d}x\\ &= \sup_{\substack{a\in A\\ z\in B}} \int_{\mathcal{X}} \chi(g_z(x)\le 0) f_a^X(x)\,\mathrm{d}x\end{aligned}$$

for strong independence and to $\overline{q}$ and the upper probability for epistemic irrelevance:

$$\begin{aligned}\overline{q}(x) &= \sup_{z\in B} \int_{\mathcal{Z}} \chi(h(x,\eta)\le 0)\,\delta_z(\eta)\,\mathrm{d}\eta \qquad (40)\\ &= \sup_{z\in B} \chi(g_z(x)\le 0) = \chi(\underline{g}(x)\le 0),\end{aligned}$$

$$\overline{p}_f^{X\not\to Z} = \sup_{a\in A}\int_{\mathcal{X}} \overline{q}(x) f_a^X(x)\,\mathrm{d}x = \sup_{a\in A}\int_{\mathcal{X}} \chi(\underline{g}(x)\le 0) f_a^X(x)\,\mathrm{d}x. \tag{41}$$

As an example we use $h(x,z) = g(x+z)$ with $z \in B = [0,2]$ moving g to the left and the limit state function g defined in the previous section. In Fig. 2 the set G, the functions $\overline{q}$, $\underline{q}$ and the upper and lower probability distribution functions $\overline{F}^{Y|x}$ and $\underline{F}^{Y|x}$ at $x = 20$ are depicted, see also Sec. 4.

Uncertainty of x modelled by an interval A:
Here we have to set $f_a^X := \delta_x$ and $a := x$. In this case the results for strong independence and epistemic irrelevance coincide:

$$\begin{aligned}\overline{p}_f^{\mathrm{S}} &= \sup_{\substack{x\in A\\ z\in B}} p_f(x,z;\delta_x,\delta_z) = \sup_{\substack{x\in A\\ z\in B}} \int_{\mathcal{X}} \chi(h(\xi,z)\le 0)\delta_x(\xi)\,\mathrm{d}\xi\\ &= \sup_{\substack{x\in A\\ z\in B}} \chi(g_z(x)\le 0) = \sup_{x\in A}\chi(\underline{g}(x)\le 0), \qquad (42)\end{aligned}$$

$$\begin{aligned}\overline{p}_f^{X\not\to Z} &= \sup_{x\in A}\int_{\mathcal{X}} \overline{q}(\xi)\delta_x(\xi)\,\mathrm{d}\xi = \sup_{x\in A}\overline{q}(x) \qquad (43)\\ &= \sup_{x\in A}\chi(\underline{g}(x)\le 0)\end{aligned}$$

because only one single $x \in A$ is used at the same time in the formulas.

We obtain the upper probabilities for our example by means of

$$\overline{p}_f^{\mathrm{S}} = \overline{p}_f^{X\not\to Z} = \sup_{x\in A}\chi(\underline{g}(x)\le 0) = \chi(\underline{g}(A)\cap(-\infty,0]\ne\varnothing) \tag{44}$$

where $g(A) = \big[\min_{x\in A} g(x), \max_{x\in A} g(x)\big] = [\underline{g(A)}, \overline{g(A)}]$ is the image of A under a function g. For the computation of $\chi(g(A)\cap(-\infty,0]\ne\varnothing)$ it is sufficient to know the lower bound $\underline{g(A)}$ of the image $g(A)$:

$$\chi(g(A)\cap(-\infty,0]\ne\varnothing) = \chi(\underline{g(A)}\le 0). \tag{45}$$

Since in our example all g_z and $\underline{g}$ are a concave functions we have $\underline{g(A)} = \min(\underline{g}(\underline{a}), \underline{g}(\overline{a})) = 0.2763$ for the interval $A = [\underline{a}, \overline{a}] = [20, 30]$ and therefore the upper probability of failure $\overline{p}_f^{\mathrm{S}} = \overline{p}_f^{X\not\to Z} = \chi(0.2763 \le 0) = 0$.

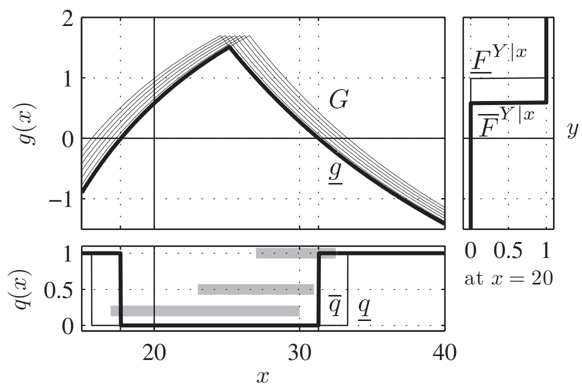

Figure 2: Set G of limit state functions g_z, lower envelope $\underline{g}$; $\underline{q}(x) = \underline{F}^{Y|x}(0)$, $\overline{q}(x) = \overline{F}^{Y|x}(0)$, focal sets of random set $\mathscr{A}$ (gray bars); $\underline{F}^{Y|x}$ and $\overline{F}^{Y|x}$ at $x = 20$.

Uncertainty of x modelled by a random set $\mathscr{A}$:
First we do some preliminary work replacing in Eqs. (25), (26), (30) and (31) the density functions by Dirac measures:

$$\overline{p}_f^{\mathrm{S}} = \sup_{\substack{x_r\in A_r,\, r=1,\ldots,|\mathscr{A}|\\ z_s\in B_s,\, s=1,\ldots,|\mathscr{B}|}} \sum_{i=1}^{|\mathscr{A}|}\sum_{j=1}^{|\mathscr{B}|} m_{\mathscr{A}}(A_i)m_{\mathscr{B}}(B_j)p_f(x_i,z_j) \tag{46}$$

$$\overline{p}_f^{X \not\to Z} = \sum_{i=1}^{|\mathscr{A}|} m_{\mathscr{A}}(A_i) \sup_{x \in A_i} \int_{\mathfrak{X}} \sum_{j=1}^{|\mathscr{B}|} m_{\mathscr{B}}(B_j) \cdot \tag{47}$$

$$\cdot \sup_{z \in B_j} \int_{\mathfrak{Z}} \chi(h(\xi, \eta) \leq 0) \delta_z(\eta) \, \mathrm{d}\eta \, \delta_x(\xi) \, \mathrm{d}\xi$$

$$= \sum_{i=1}^{|\mathscr{A}|} m_{\mathscr{A}}(A_i) \sup_{x \in A_i} \sum_{j=1}^{|\mathscr{B}|} m_{\mathscr{B}}(B_j) \sup_{z \in B_j} \chi(h(x,z) \leq 0)$$

$$= \sum_{i=1}^{|\mathscr{A}|} m_{\mathscr{A}}(A_i) \sup_{x \in A_i} \overline{q}(x)$$

$$\overline{q}(x) = \sum_{j=1}^{|\mathscr{B}|} m_{\mathscr{B}}(B_j) \sup_{z \in B_j} \chi(h(x,z) \leq 0), \tag{48}$$

$$\overline{p}_f^{\mathrm{RS}} = \overline{p}_f^{\mathrm{R}, X \not\to Z} = \sum_{i=1}^{|\mathscr{A}|} \sum_{j=1}^{|\mathscr{B}|} m_{\mathscr{A}}(A_i) m_{\mathscr{B}}(B_j) \sup_{\substack{x \in A_i \\ z \in B_j}} p_f(x,z) \tag{49}$$

where $p_f(x,z) = \chi(h(x,z) \leq 0)$ and where $\overline{p}_f^{\mathrm{RS}} = \overline{p}_f^{\mathrm{R},X \not\to Z}$ coincides for Dirac measures as already mentioned. While these equations are needed in the next section we further use here that we have only one set B with weight 1 which leads to the following simplified versions:

$$\overline{p}_f^{\mathrm{S}} = \sup_{\substack{x_r \in A_r, r=1,\ldots,|\mathscr{A}| \\ z \in B}} \sum_{i=1}^{|\mathscr{A}|} m_{\mathscr{A}}(A_i) p_f(x_i, z), \tag{50}$$

$$\overline{p}_f^{X \not\to Z} = \sum_{i=1}^{|\mathscr{A}|} m_{\mathscr{A}}(A_i) \sup_{x \in A_i} \sup_{z \in B} \chi(h(x,z) \leq 0), \tag{51}$$

$$\overline{p}_f^{\mathrm{RS}} = \overline{p}_f^{\mathrm{R},X \not\to Z} = \sum_{i=1}^{|\mathscr{A}|} m_{\mathscr{A}}(A_i) \sup_{\substack{x \in A_i \\ z \in B}} p_f(x,z) \tag{52}$$

where $\overline{p}_f^{X \not\to Z} = \overline{p}_f^{\mathrm{R},X \not\to Z}$ because $p_f(x,z) = \chi(h(x,z) \leq 0)$.

The difference between $\overline{p}_f^{\mathrm{S}}$ and $\overline{p}_f^{X \not\to Z}$ is that there is a single z used for all x_r together in case of strong independence while for epistemic irrelevance z can be chosen for each x_r separately. The numerical results are obtained by

$$\overline{p}_f^{\mathrm{S}} = \sup_{z \in B} \sum_{i=1}^{|\mathscr{A}|} m_{\mathscr{A}}(A_i) \sup_{x \in A_i} \chi(g_z(x) \leq 0) \tag{53}$$

$$= \sup_{z \in B} \mathrm{Pl}_{\mathscr{A}}(g_z(x) \leq 0) = 0.5$$

and

$$\overline{p}_f^{X \not\to Z} = \sum_{i=1}^{|\mathscr{A}|} m_{\mathscr{A}}(A_i) \sup_{x \in A_i} \chi(\underline{g}(x) \leq 0) \tag{54}$$

$$= \mathrm{Pl}_{\mathscr{A}}(\underline{g}(x) \leq 0) = 0.2 \cdot 1 + 0.3 \cdot 0 + 0.5 \cdot 1 = 0.7,$$

cf. Eqs. (50), (51).

We have always $\overline{p}_f^{\mathrm{S}} = \overline{p}_f^{X \not\to Z}$ if $\underline{g} \in G$. This holds in the case where $h(x,z) = g(x) + z$, $z \in B = [\underline{b}, \overline{b}]$ $\underline{g}(x) = g(x) + \underline{b}$ and $\overline{g}(x) = g(x) + \overline{b}$.

6.2 Random sets of parameterized limit state functions

Modelling the uncertainty of the limit state function:
For modelling the uncertainty of the parameter z we use a random set $\mathscr{B}$ given by the following focal sets B_j and weights $m_{\mathscr{B}}(B_j)$:

$$B_1 = [-0.9, 1.3], \quad m_{\mathscr{B}}(B_1) = 0.1,$$
$$B_2 = [-0.6, 0.9], \quad m_{\mathscr{B}}(B_2) = 0.3,$$
$$B_3 = [-0.4, 0.6], \quad m_{\mathscr{B}}(B_3) = 0.4,$$
$$B_4 = [-0.2, 0.4], \quad m_{\mathscr{B}}(B_4) = 0.2.$$

In the view of Sec. 4 we define a random set $\mathscr{G}$ of limit state functions by the focal sets $G_j = \{g_z : z \in B_j\}$ and weights $m_{\mathscr{G}}(G_j) = m_{\mathscr{B}}(B_j)$. At a point $x \in \mathfrak{X}$ we have then a random set $\mathscr{G}(x)$ with focal sets $G_j(x) = \{g(x) : g \in G_j\}$ and the same weights, which describes the output of the uncertain limit state function.

The function $\overline{q}$ is obtained by

$$\overline{q}(x) = \overline{F}^{Y|x}(0) = \sum_{j=1}^{|\mathscr{B}|} m_{\mathscr{B}}(B_j) \chi(G_j(x) \cap (-\infty, 0] \neq \emptyset)$$

$$= \mathrm{Pl}_{\mathscr{G}(x)}((-\infty, 0]) \tag{55}$$

which is the plausibility measure of $(-\infty, 0]$ for the random set $\mathscr{G}(x)$ at x. The lower bound $\underline{q}$ is the belief measure at x (cf. Fig. 3):

$$\underline{q}(x) = \underline{F}^{Y|x}(0) = \sum_{j=1}^{|\mathscr{B}|} m_{\mathscr{B}}(B_j) \chi(G_j(x) \subseteq (-\infty, 0])$$

$$= \mathrm{Bel}_{\mathscr{G}(x)}((-\infty, 0]). \tag{56}$$

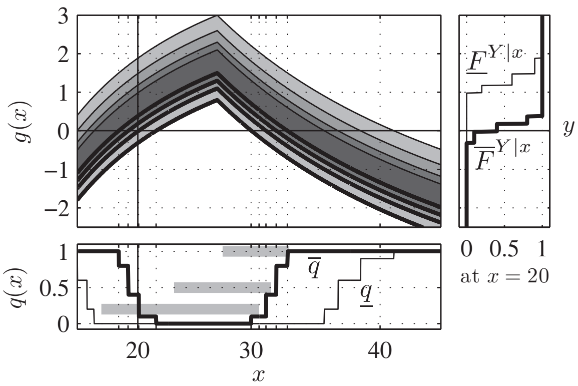

Figure 3: Random set $\mathscr{G}$, lower envelopes g^j; $\underline{q}(x) = \underline{F}^{Y|x}(0)$ and $\overline{q}(x) = \overline{F}^{Y|x}(0)$, focal sets of random set $\mathscr{A}$ (gray bars); $\underline{F}^{Y|x}$ and $\overline{F}^{Y|x}$ at $x = 20$.

Uncertainty of x modelled by a random set $\mathscr{A}$:
We consider now the special case where $h(x,z)$ is given by $g_z(x) = h(x,z) = g(x) + z$ resulting in the uncertain limit state function depicted in Fig. 3.

Then it holds for the lower envelopes $\underline{g}^j$ of the focal sets G_j that $\underline{g}^j \in G_j$. It is clear that we can reduce the focal sets G_j to their lower envelopes which leads to a discrete set of limit state functions eqipped with a probability distribution induced by the weights of the focal sets G_j. But then there is only one single probability distribution and therefore no possibility of choice which leads to $\overline{p}_f^{\mathrm{S}} = \overline{p}_f^{X \not\perp Z}$. Further we have $\overline{p}_f^{\mathrm{S}} = \overline{p}_f^{\mathrm{RS}}$ because of the ordering of the four lower envelopes ($\underline{g}^1 \le \underline{g}^2 \le \underline{g}^3 \le \underline{g}^4$, see Fig. 3).

In the following the upper probabilities $\overline{p}_f^{\mathrm{RS}} = \overline{p}_f^{\mathrm{R},X \not\perp Z}$ and $\overline{p}_f^{X \not\perp Z}$ are computed: Since in our example the results coincide for all notions of independence we have the possibility to choose between two methods for the upper probability of failure where either discontinuous or continuous optimization problems involved: For the upper probability $\overline{p}_f^{X \not\perp Z}$ in case of epistemic irrelevance we have to solve $|\mathscr{A}|$ discontinuous ($\overline{q}$ is discontinuous) optimization problems

$$\overline{p}_f^{X \not\perp Z} = \sum_{i=1}^{|\mathscr{A}|} m_{\mathscr{A}}(A_i) \sup_{x \in A_i} \overline{q}(x) \tag{57}$$
$$= \sum_{i=1}^{|\mathscr{A}|} m_{\mathscr{A}}(A_i) \sup_{x \in A_i} \sum_{j=1}^{|\mathscr{B}|} m_{\mathscr{B}}(B_j) \chi(\underline{g}^j(x) \le 0)$$
$$= 0.2 \cdot 1.0 + 0.3 \cdot 0.4 + 0.5 \cdot 1.0 = 0.82$$

and for the upper probability in case of random set independence $|\mathscr{A}| \cdot |\mathscr{B}|$ continuous one (g is continuous):

$$\overline{p}_f^{\mathrm{RS}} = \overline{p}_f^{\mathrm{R},X \not\perp Z} = \sum_{i,j} m_{\mathscr{A}}(A_i) m_{\mathscr{B}}(B_j) \sup_{\substack{x \in A_i \\ z \in B_j}} p_f(x,z) \tag{58}$$
$$= \sum_{j=1}^{|\mathscr{B}|} m_{\mathscr{B}}(B_j) \sum_{i=1}^{|\mathscr{A}|} m_{\mathscr{A}}(A_i) \chi(\underline{\underline{g}^j(A_i)} \le 0)$$
$$= \sum_{j=1}^{|\mathscr{B}|} m_{\mathscr{B}}(B_j) \mathrm{Pl}_{\mathscr{A}}(\underline{g}^j(x) \le 0)$$
$$= 0.1 \cdot 1.0 + 0.3 \cdot 1.0 + 0.4 \cdot 0.7 + 0.2 \cdot 0.7 = 0.82.$$

6.3 Random limit state functions

We have again $h(x,z) = g(x) + z$ and model the uncertainty of the parameter z by a Gaussian distribution (density f_b^Z) with parameters $b = (\mu, \sigma)$.

Let us start with deterministic parameters, say $b = (0, 0.5)$, which leads to $\overline{p}_f^{\mathrm{S}} = \overline{p}_f^{X \not\perp Z}$. Using the notation of Sec. 4 we have $Y_{|x} = g(x) + Z$ with random variable $Z \sim \mathcal{N}(\mu, \sigma^2)$ and conditional random variable $Y_{|x} \sim \mathcal{N}(g(x) + \mu, \sigma^2)$ given the basic variable x. The function q is obtained by

$$q(x) = \int_{\mathcal{Z}} \chi(g(x) + z \le 0) f^Z_{(0,0.5)}(z)\, \mathrm{d}z = F^{Y|x}_{(g(x),0.5)}(0) \tag{59}$$

where $F^{Y|x}$ is the probability distribution function of $Y_{|x}$, cf. Sec. 4 and Fig. 4.

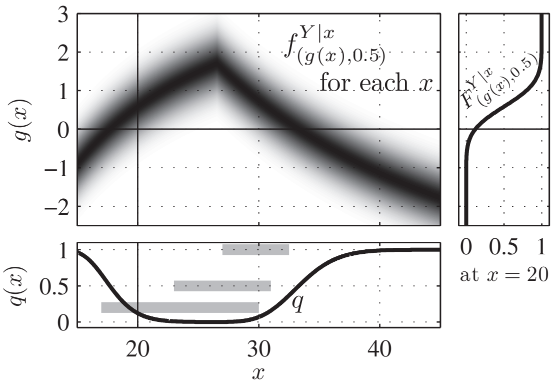

Figure 4: Uncertain limit state function $g(x) + z$ where the uncertainty of z is described by a Gaussian distribution with $\mu = 0$ and $\sigma = 0.5$; $q(x) = F^{Y|x}_{(g(x),0.5)}(0)$, focal sets of random set $\mathscr{A}$; $F^{Y|x}_{(g(x),0.5)}$ at $x = 20$.

Uncertainty of x modelled by an interval $A = [20, 30]$: We obtain

$$\overline{p}_f^{\mathrm{S}} = \overline{p}_f^{X \not\perp Z} = \sup_{x \in A} p_f(x, b; \delta_x, f_b^Z) = \sup_{x \in A} q(x) = 0.1222 \tag{60}$$

using Eqs. (19) and (20).

Uncertainty of x modelled by a random set: We get

$$\overline{p}_f^{\mathrm{S}} = \overline{p}_f^{\mathrm{R},X \not\perp Z} = \overline{p}_f^{\mathrm{RS}} = \overline{p}_f^{X \not\perp Z} = \sum_{i=1}^{|\mathscr{A}|} m_{\mathscr{A}}(A_i) \sup_{x \in A_i} q(x) \tag{61}$$
$$= 0.2 \cdot 0.6604 + 0.3 \cdot 0.1576 + 0.5 \cdot 0.3682 = 0.3635$$

for the random set $\mathscr{A}$ given in Sec. 5.2 using very simplified versions of Eqs. (26), (30) and (31).

Uncertainty of x modelled by a single probability distribution: For a Gaussian distribution (density f_a^X) with deterministic parameters $a = (\mu, \sigma) = (34, 1)$ we get the result

$$p_f((34,1),(0,0.5), f_a^X, f_b^Z) = \tag{62}$$
$$= \int_{\mathcal{X}} \int_{\mathcal{Z}} \chi(g(x) + z \le 0) f^Z_{(0,0.5)}(z)\, \mathrm{d}z\, f^X_{(34,1)}(x)\, \mathrm{d}x$$
$$= \int_{\mathcal{X}} q(x) f^X_{(34,1)}(x)\, \mathrm{d}x = 0.5976.$$

Uncertainty of b modelled by a set B: Let the set B for the parameters b in f_b^Z given by

$$[\underline{\mu}, \overline{\mu}] \times [\underline{\sigma}, \overline{\sigma}] = [-0.3, 0.3] \times [0.2, 0.6].$$

The function $\overline{q}$ and the corresponding lower bound $\underline{q}$ are obtained here by

$$\overline{q}(x) = \overline{F}^{Y|x}(0) = \begin{cases} F^{Y|x}_{(g(x)+\underline{\mu},\overline{\sigma})}(0) & \text{if } g(x) + \underline{\mu} > 0, \\ F^{Y|x}_{(g(x)+\underline{\mu},\underline{\sigma})}(0) & \text{if } g(x) + \underline{\mu} \le 0 \end{cases} \tag{63}$$

and

$$\underline{q}(x) = \underline{F}^{Y|x}(0) = \begin{cases} F^{Y|x}_{(g(x)+\overline{\mu},\underline{\sigma})}(0) & \text{if } g(x)+\overline{\mu} > 0, \\ F^{Y|x}_{(g(x)+\overline{\mu},\overline{\sigma})}(0) & \text{if } g(x)+\overline{\mu} \leq 0. \end{cases} \tag{64}$$

In Fig. 5 the densities $f^{Y|x}_{(g(x)+\underline{\mu},\overline{\sigma})}$ and $f^{Y|x}_{(g(x)+\underline{\mu},\underline{\sigma})}$ resulting in $\overline{q}$ are depicted as well as the functions $\overline{q}$, $\underline{q}$ and the upper and lower distribution functions $\overline{F}^{Y|x}$ and $\underline{F}^{Y|x}$ at $x = 20$. The numerical results for uncertain x as above (set A, random set $\mathscr{A}$, probability distribution) can be obtained in case of epistemic irrelevance by simply replacing q by $\overline{q}$ in the Eqs. (60), (61) and (62). The results for $\overline{p}_f^{X \not\to Z}$ are 0.3192 for the set A, 0.6817 for the random set $\mathscr{A}$ and 0.9387 for the Gaussian distribution.

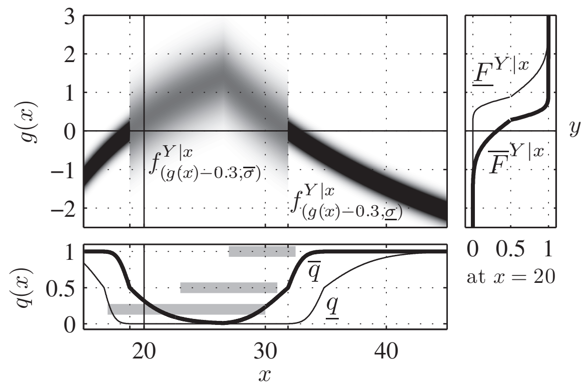

Figure 5: Uncertain limit state function $g(x)+z$ where the uncertainty of z is modelled by a set of Gaussian distributions.

Conclusion

To model uncertainties in limit state functions g we extended g depending on basic variables x to functions h by adding additional parameters z and introduced a function $p_f(a,b)$ for the probability of failure. This function provides an interface for controlling the parameters a and b of the probability density functions f_a^X and f_b^Z used for modelling the uncertainty of the basic variables x and the new additional parameters z. In a next step the two parameters a and b were assumed to be uncertain using sets or random sets to model their uncertainty resulting in sets of proability measures for x and z. In this context we discussed several notions of independence, gave computational formulas for different cases of uncertainty models exemplified by a simple engineering example and addressed visualization methods and alternative approaches as well.

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