Two for the Price of *One*: Info-Gap Robustness of the One-Test Algorithm

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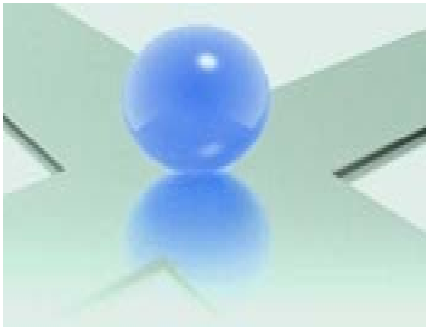

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1 *Highlights*

§ **Binary decisions, expensive tests:**

- **Airplane designs:**
  Lockheed or Raytheon?
- **Pollution control:**
  Tax or cap-and-trade?
- **Medical intervention:**
  Surgery or pharmaceuticals?
- **Military budget:**
  Tanks or intelligence?
- **Energy supply:**
  Nuclear or fossil?

2 *Two Systems, One Test*

§ **Two systems, with qualities $x_1 \neq x_2 \in \Re$.**
Bigger is better.

§ **No prior knowledge?**

- **Flip a fair coin.**
- **50/50 chance of success.**

§ **One system tested: quality $x_{\rm r}$.**

- **Enhanced chance of success?**
- **Which system to use?**

[0] lectures\talks\lib\two-systems02ten-min.tex, 19.6.2011. See 'Problem Set on Info-Gap Uncertainty', \lectures\risk\homework\ps1_rk.tex, #10. Yakov Ben-Haim, 2011, Two for the price of one: Info-gap robustness of the 1-test algorithm, ISIPTA2011, 25–28 July 2011, Innsbruck, Austria.

§ Algorithm for choosing a system:

- $q(y)$ is **any** pdf: $q(y) > 0$ for all y.
- Draw y from $q(y)$.
- If $y \geq x_{\mathrm{r}}$ then **choose un-tested** system.
- If $y < x_{\mathrm{r}}$ then **choose tested** system.

§ Probability of success, $P_{\mathrm{s}}(q)$:

Probability of choosing larger x_i.

§ Theorem:

If tested system chosen with prob. 0.5,

then $P_{\mathrm{s}}(q) > 0.5$.

3 *Robustness of Two Systems, One Test*

§ **How to choose $q(y)$?**

Can we beat $P_{\rm s}(q) > 0.5$?

§ **If we know $p(x_i)$ then $P_{\rm s} = 0.75$.**

Can we achieve $P_{\rm s}(q) = 0.75$?

§ Info-gap robust-satisficing:

- **Our guess:** $x \sim \widetilde{p}(x)$.
- $\widetilde{p}(x)$ **highly uncertain.**
- **Choose** $q(y)$ **to robust satisfice:**
  - **Satisfy** $P_{\mathrm{s}} \geq P_{\mathrm{c}}$.
  - **Maximize robustness to uncertain** $\widetilde{p}$.

§ Info-gap model for uncertain $\widetilde{p}(x)$: $\mathcal{U}(h)$.

- **Nesting:** $h < h' \implies \mathcal{U}(h) \subseteq \mathcal{U}(h')$.
- **Contraction:** $\mathcal{U}(0) = \{\widetilde{p}\}$.
- h **is unbounded horizon of uncertainty.**

§ Robustness, $\widehat{h}(q, P_{\mathrm{c}})$:

Maximum tolerable uncertainty.

$$\widehat{h}(q, P_{\mathrm{c}}) = \max \left\{ h : \ \left(\min_{p \in \mathcal{U}(h)} P_{\mathrm{s}}(q|p) \right) \geq P_{\mathrm{c}} \right\}$$

§ Example:

- **Estimated pdf:** $\widetilde{p}(x) = \widetilde{\lambda}\mathrm{e}^{-\widetilde{\lambda}x}$.
- **Decision pdf:** $q(y) = \gamma\mathrm{e}^{-\gamma y}$.
- **Prob of success:** $P_{\mathrm{s}}(q|\widetilde{p}) > 0.5$
- **Putative optimal choice:**

$$\begin{aligned} \gamma^\star &= \arg\max_{\gamma} P_{\mathrm{s}}(q|\widetilde{p}) \\ &= \widetilde{\lambda}\sqrt{2} \end{aligned}$$

- **E.g.,** $\widetilde{\lambda} = 1$**:** $P_{\mathrm{s}}(q|\widetilde{p}) = 0.67 \gg 0.5$
- **Robust to uncertainty in** $\widetilde{p}(x)$**???**

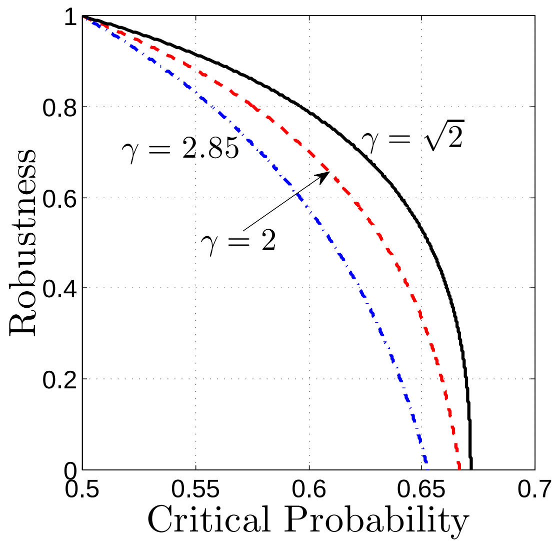

Figure 1: Robustness curves with $\widetilde{\lambda} = 1$.

§ Trade off: robustness vs prob. of success.

§ Zeroing:

Estimated prob of success: no robustness.

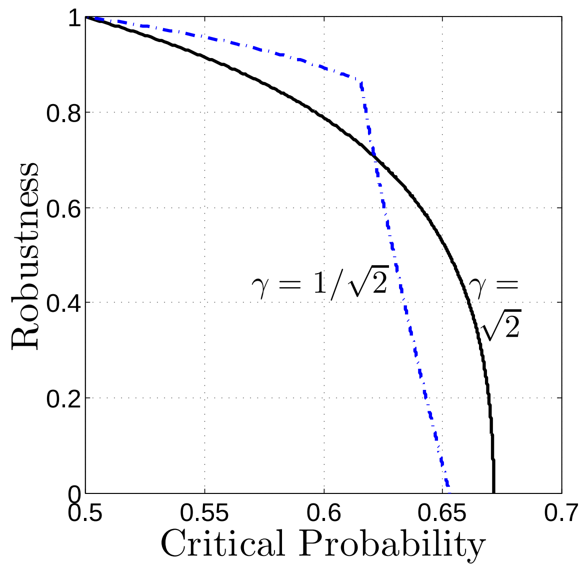

Figure 2: Robustness curves with $\widetilde{\lambda} = 1$.

§ **Trade off: robustness vs prob. of success.**

§ **Zeroing: no robustness of estimate.**

§ **Preference reversal.**

- $\gamma = \sqrt{2}$ **more robust for** $P_c > 0.62$**.**
- $\gamma = 1/\sqrt{2}$ **more robust for** $P_c < 0.62$**.**

4 *Three Systems, Two Tests*

§ **3 Systems with qualities:**

$$x_1 < x_2 < x_3$$

§ **Test two systems with revealed attributes:**

$$r_1 < r_2$$

§ **Goal: Exclude worst system.**

§ **Blind probability of success:** $\frac{1}{3}$

§ **Algorithm:**

- $q(y)$ **any non-zero pdf.**
- **Draw** y **from** $q(q)$**.**
- **If** $y < r_1$ **choose 2 tested systems.**
- **If** $r_1 \leq y$ **choose** r_2 **and untested system.**

§ **Theorem:**

If tested systems chosen with equal prob.

then $P_{\rm s}(q) > \frac{1}{3}$**.**

5 *Three Systems, One Test*

§ **3 Systems with qualities:**

$$x_1 < x_2 < x_3$$

§ **Test one system with revealed attribute r.**

§ **Goal: Select best system.**

§ **Blind probability of success: $\frac{1}{3}$**

§ **Algorithm:**

- $q(y)$ **any non-zero pdf.**
- **Draw y from $q(q)$.**
- **If $y \le r$ choose tested system.**
- **If $r < y$ choose equi-prob from untested.**

§ **Theorem:**

If tested system chosen with equal prob.

then $P_{\rm s}(q) > \frac{1}{3}$.

6 n Systems, m Tests

§ **Hypothesized generalization to n systems, m tests.**

7 *Extensions*

§ Multiple attributes.

§ Adaptive testing.

§ Best possible probability of success.

§ Robust satisficing.

§ Opportune windfalling.