Conglomerable natural extension

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Research group

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Research interests:

- Imprecise probabilities: coherent lower previsions, non-additive measures, random sets, independence.
- Fuzzy preference structures.
- Divergence measures.

Where we are

Where we are

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Introduction

Within subjective probability, conglomerability means that if we accept a transaction conditional on any element of a given partition, then we should also accept it in general.

Although intuitive, it has some undesirable properties, and it is rejected by some authors such as de Finetti and Williams.

Our goal in this paper is to investigate the most conservative extension of some assessments that satisfies conglomerability.

Outline

1. Introduction to conglomerability.
2. Conglomerable natural extension for sets of gambles.
3. Conglomerable natural extension of lower previsions.
4. The case of several partitions.
5. Conclusions and open problems.

Conglomerability for sets of gambles

A set of gambles $\mathcal{R}$ is called coherent when:

(D1) $f \gneq 0 \Rightarrow f \in \mathcal{R}$;

(D2) $0 \notin \mathcal{R}$;

(D3) $f \in \mathcal{R}, \lambda > 0 \Rightarrow \lambda f \in \mathcal{R}$;

(D4) $f, g \in \mathcal{R} \Rightarrow f + g \in \mathcal{R}$.

Given a partition $\mathcal{B}$ of Ω, $\mathcal{R}$ is called $\mathcal{B}$-conglomerable when

(D5) $f \neq 0$ and $Bf \in \mathcal{R} \cup \{0\} \forall B \in \mathcal{B} \Rightarrow f \in \mathcal{R}$.

Conglomerability for lower previsions

A lower prevision $\underline{P}$ on $\mathcal{L}$ is called coherent when:

(C1) $\underline{P}(f) \geq \inf f$ for all $f \in \mathcal{L}$;

(C2) $\underline{P}(\lambda f) = \lambda \underline{P}(f)$ for all $f \in \mathcal{L}$ and $\lambda > 0$;

(C3) $\underline{P}(f+g) \geq \underline{P}(f) + \underline{P}(g)$ for all $f, g \in \mathcal{L}$.

$\underline{P}$ is called $\mathcal{B}$-conglomerable when $(B_n)_n$ pairwise disjoint, $\underline{P}(B_n) > 0$ and $\underline{P}(B_n f) \geq 0 \ \forall n \Rightarrow \underline{P}(\sum_n B_n f) \geq 0$.

Relationship

If we make the correspondence

$$\mathcal{R} \leftrightarrow \underline{P}(f) := \sup\{\mu : f - \mu \in \mathcal{R}\}$$

then $\mathcal{R}$ coherent $\Leftrightarrow$ $\underline{P}$ coherent.

- However, the conglomerability condition for sets of desirable gambles is stronger than the one for lower previsions!

Conglomerable natural extension of a set of gambles

Let $\mathcal{R}$ be a coherent set of gambles. The smallest superset $\mathcal{F}$ that satisfies (D1)–(D5) with respect to a fixed partition $\mathcal{B}$ is called the $\mathcal{B}$-conglomerable natural extension of $\mathcal{R}$.

- $\mathcal{F}$ may not exist.
- Its existence does not imply the existence of a conglomerable half-space of gambles including $\mathcal{R}$ $\rightarrow$ we don't have envelope like results.

Approximation by a sequence

Let us define the following sequence:

$$\mathcal{R}^* := \{f \neq 0 \colon (\forall B \in \mathcal{B}) Bf \in \mathcal{R} \cup \{0\}\}$$
$$\mathcal{E}_1 := \mathcal{R} \oplus \mathcal{R}^*$$

and for all $n \geq 2$:

$$\mathcal{E}^*_{n-1} := \{f \neq 0 \colon (\forall B \in \mathcal{B}) Bf \in \mathcal{E}_{n-1} \cup \{0\}\}$$
$$\mathcal{E}_n := \mathcal{E}_{n-1} \oplus \mathcal{E}^*_{n-1}.$$

- $\mathcal{E}_n \subseteq \mathcal{F}$, and it need not be $\mathcal{F} = \mathcal{E}_1$.

Conglomerable natural extension of a lower prevision

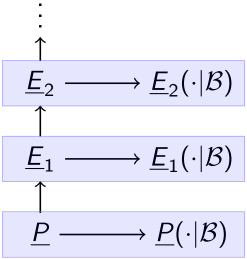
Similarly, given a coherent lower prevision $\underline{P}$ on $\mathcal{L}$ its $\mathcal{B}$-conglomerable natural extension is the smallest coherent lower prevision $\underline{F}$ that dominates $\underline{P}$ and is $\mathcal{B}$-conglomerable.

Let $\underline{P}(\cdot|\mathcal{B})$ be the conditional natural extension of $\underline{P}$, and $\underline{E}$ the natural extension of $\underline{P}, \underline{P}(\cdot|\mathcal{B})$.

- $\underline{E} \leq \underline{F}$, but they do not coincide in general.

Approximation by a sequence

More generally, we can consider the construction:



where $\rightarrow$ applies the conditional natural extension and $\uparrow$ the unconditional natural extension.

- $\underline{E}_n \leq \underline{F}\ \forall n$, and $\underline{E}_n \neq \underline{E}_{n+1}$ unless $\underline{E}_n = \underline{F}$.

Connection between the two approaches

Let $\mathcal{R}$ be a coherent set of desirable gambles and $\underline{P}$ its associated coherent lower previsions. Consider the approximating sequences $(\mathcal{E}_n)_n$ and $(\underline{E}_n)_n$ of their conglomerable natural extensions.

- $\mathcal{E}_1 = \mathcal{F} \Rightarrow \underline{E}_1 = \underline{F}$, *but the converse is not true!*
- Let $(\underline{P}_n)_n$ be the sequence of coherent lower previsions associated to $(\mathcal{E}_n)_n$. Then $\underline{E}_n \leq \underline{P}_n$ for every n, and they coincide if $\underline{P}(B) > 0 \ \forall B \in \mathcal{B}$.

The case of several partitions

Consider now several partitions $\mathcal{B}_1, \dots, \mathcal{B}_n$ of Ω.

- Conglomerability with respect to each of $\mathcal{B}_1, \dots, \mathcal{B}_n$ is equivalent to the conglomerability with respect to all the partitions that can be derived from them.
- Similarly to the marginal extension theorem, when the partitions are increasingly finer, we can compute the conglomerable natural extension in one step.
- In the case of coherent lower previsions, it is related to weak coherence.

Conclusions

- Walley's study of conditional coherence is based on the notion of conglomerability, but the natural extension does not necessarily satisfy this condition, even if the conglomerable natural extension exists.
- In a number of particular cases, the conglomerable natural extension coincides with the natural extension; we can also approximate it by means of a sequence.

Open problems

- Do the sequences $(\mathcal{E}_n)_n$ and $(\underline{E}_n)_n$ always stabilise in a finite number of steps?
- Is it $\mathcal{F} = \cup_n \mathcal{E}_n$ and $\underline{F} = \lim_n \underline{E}_n$?
- Determine if the definition of natural extension of conditional lower previsions can be modified to encompass conglomerability.