Non-conflicting and Conflicting Parts of Belief Functions

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- Conclusion

Introduction

Let us suppose normalized BFs on finite frames.

conjuntive combination of BFs
conflicting belief masses (disjoint focal elements)

belief mass $\longrightarrow$ $\emptyset$ (non-normalized conjunctive rule ... $\bigcirc\!\!\!\!\cap$)
$\longrightarrow$ relocation/redistribution among some $\emptyset \neq X \subseteq \Omega$
$m_{\bigcirc\!\!\!\!\cap}(\emptyset)$ ... weight of conflict between BFs (Shafer 76)

– simple examples, which do not support this interpretation
$\times$
– $m_{\bigcirc\!\!\!\!\cap}(\emptyset)$ ... really conflicting belief masses, related to conflict

IPMU'10 : $m_{\bigcirc\!\!\!\!\cap}(\emptyset)$ — internal conflict of input BFs
— conflict between BFs

3 new approaches to conflicts were introduced there (ideas, motivations, open problems) + distingushing: difference $\times$ conflict between BFs)

analyzing properties: possibility of decomposition $Bel = Bel_0 \oplus Bel_S$
non-conflicting and conflicting part of BF Bel

Existence and uniqueness of BFs Bel_0 and Bel_S is studied here

Basic notions on belief functions

Exhaustive finite n-element frame of discernment $\Omega = \{\omega_1, \omega_2, ... \omega_n\}$, all elements ω_i are mutually exclusive. unknown actual $\omega_0 \in \Omega$

Basic belief assignment (bba) $m : \mathcal{P}(\Omega) \longrightarrow [0, 1]$, s.t. $\sum_{A \subseteq \Omega} m(A) = 1$
values ..... *basic belief masses (bbm)*, if $m(\emptyset) = 0$ ..... *normalized bba*
Belief function (BF) $Bel : \mathcal{P}(\Omega) \longrightarrow [0, 1]$, $Bel(A) = \sum_{\emptyset \neq X \subseteq A} m(X)$,
Bel uniquely corresponds to bba m and vice-versa.
Plausibility function, Commonality function $Pl, Q : \mathcal{P}(\Omega) \longrightarrow [0, 1]$,

Focal element ..... $X \subseteq \Omega$, such that $m(X) > 0$.
Bayesian Belief function (BBF): $|X| = 1$ for $m(X) > 0$, $U_2 = 0'$
U_n ... *Uniform BBF* ... $U_n(\{\omega_i\}) = \frac{1}{n}$ ($\sim$ uniform prob. distrib. on Ω)

Dempster's (conjunctive) rule of combination $\oplus$:
$(m_1 \oplus m_2)(A) = \sum_{X \cap Y = A} K m_1(X) m_2(Y)$ for $A \neq \emptyset$, $(m_1 \oplus m_2)(\emptyset) = 0$,
where $K = \frac{1}{1-\kappa}$, $\kappa = \sum_{X \cap Y = \emptyset} m_1(X) m_2(Y)$, $\textcircled{\scriptsize$\cap$}$: $K = 1$, $m(\emptyset) = \kappa$
the *disjunctive rule* $\textcircled{\scriptsize$\cup$}$, *Yager's rule* $\textcircled{\scriptsize Y}$, *Dubois-Prade's rule* $\textcircled{\scriptsize P}$, ...

indecisive (indifferent) BF: $h(Bel) = Bel \oplus U_n = U_n$, i.e., $Pl(\{\omega_i\}) = const.$
non-conflicting BF Bel: $(Bel \textcircled{\scriptsize$\cap$} Bel)(\emptyset) = 0$; conflicting BF otherwise

pignistic prob, $BetP(\omega_i)$; normalized plausib. of singletons $(Pl\_P(m))(\omega_i)$, ...

Dempster's semigroup

$\Omega_2 = \{\omega_1, \omega_2\}$

(P.Hájek & J.J.Valdés 80's/90's)

$\mathbf{D}_0 = (D_0, \oplus, 0, 0')$

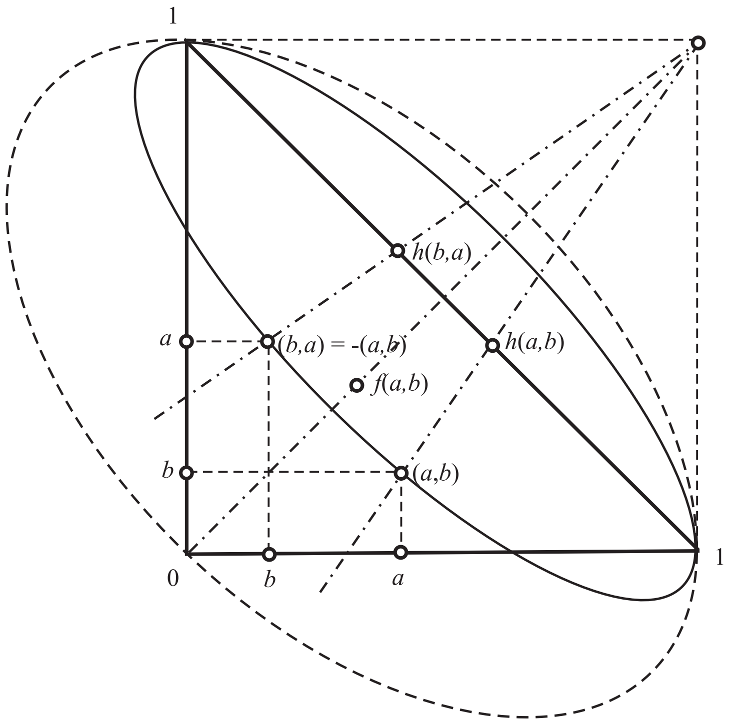
Ω_2: $m \sim (a, b) = (m(\{\omega_1\}, m(\{\omega_2\}))$ as $m(\{\omega_1, \omega_2\}) = 1 - (a + b)$,

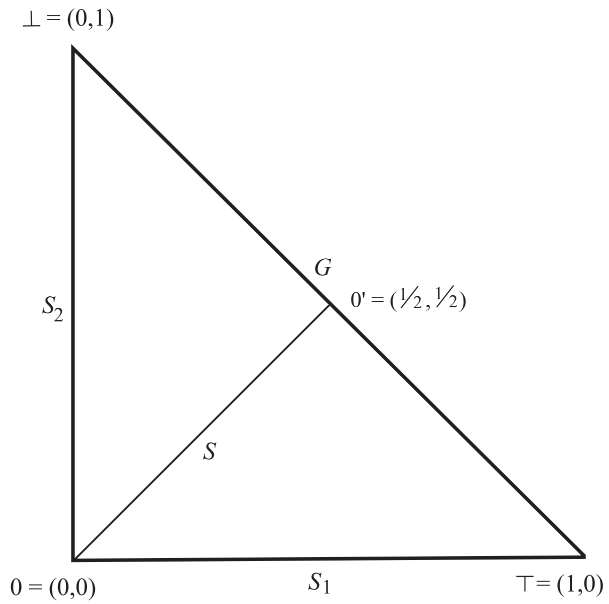
d-pairs ... $(a, b) : 0 \le a, b \le 1, a + b \le 1$

$D_0 = \{(a, b) \,|\, 0 \le a, b < 1, a + b \le 1\}$ ... set of non-extremal d-pairs

Dempster's rule $\oplus$: $(a, b) \oplus (c, d) = (1 - \frac{(1-a)(1-c)}{1-(ad+bc)}, \; 1 - \frac{(1-b)(1-d)}{1-(ad+bc)})$

(for d-pairs)





extremal d-pairs:

$\bot = (0, 1)$, $\top = (1, 0)$

VBF: $0 = (0, 0)$

$0' = U_2 = (\frac{1}{2}, \frac{1}{2})$

$h : h(a, b) = (a, b) \oplus 0'$

$- : -(a, b) = (b, a)$

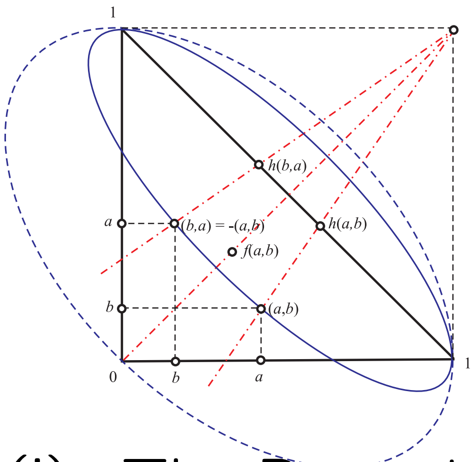
$f : f(a, b) = (a, b) \oplus -(a, b)$

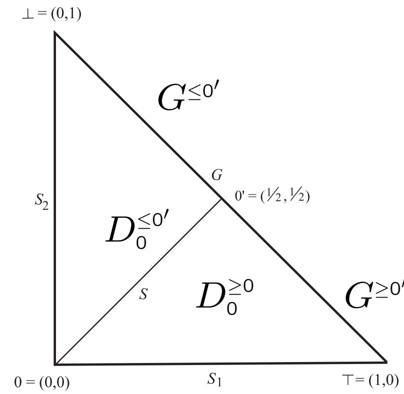
$G = \{(a, 1 - a) \,|\, 0 \le a \le 1\}$ ... Bayesian d-pairs

$S = \{(a, a) \,|\, 0 \le a \le \frac{1}{2}\}$

$S_2 = \{(0, a) \,|\, 0 \le a \le 1\}$, $S_1 = \{(a, 0) \,|\, 0 \le a \le 1\}$, ... simple d-pairs

Dempster's semigroup (cont.)





$(a,b) \leq (c,d)$ iff
$[\,h_1(a,b) < h_1(c,d)$
or $h_1(a,b) = h_1(c,d)$ and $a \leq c]$,
where $h(a,b) = (h_1(a,b), h_2(a,b))$,
thus $h_1(a,b) = \frac{1-b}{2-a-b}$; $\quad D_0^{\leq 0'}$, $D_0^{\geq 0}$.

(i) The Dempster's semigroup $\mathbf{D}_0$ with the relation $\leq$ is an ordered commutative (Abelian) semigroup with the neutral element 0; $0'$ is the only non-zero idempotent of $\mathbf{D}_0$.

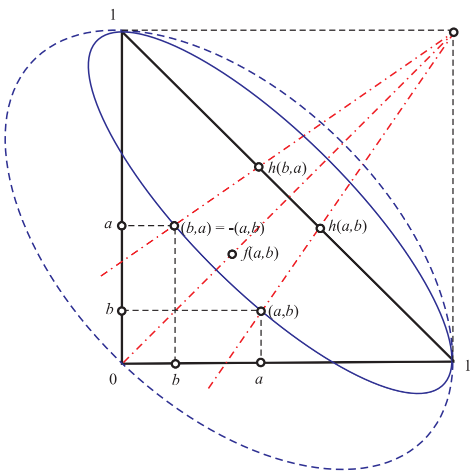
(ii) $\mathbf{G} = (G, \oplus, -, 0', \leq)$ is an ordered Abelian group, isomorhpic to the group of reals with the usual ordering. $G^{\leq 0'}$ and $G^{\geq 0'}$ ... its negative and pos. cones.

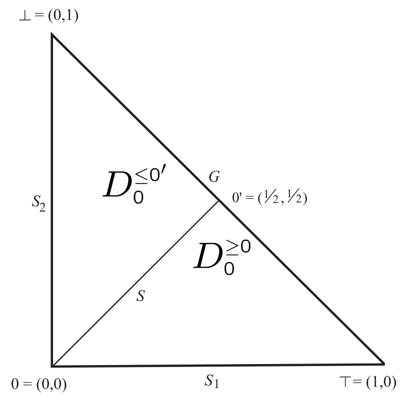
(iii) The sets S, S_1, S_2 with the operation $\oplus$ and the ordering $\leq$ form ordered commutative semigroups with neutral element 0, all are isomorphic to the positive cone of the additive group of reals.

(iv) h is ordered homomorphism: $(D_0, \oplus, -, 0, 0', \leq) \longrightarrow (G, \oplus, -, 0', \leq)$;
$h(Bel) = Bel \oplus 0' = Pl\text{-}P(Bel)$, i.e., normalized plausibility probabilistic transf.

(v) f is homomorphism: $(D_0, \oplus, -, 0, 0') \longrightarrow (S, \oplus, -, 0)$; (not ordered).

Dempster's semigroup (cont.)





Let us denote
$h^{-1}(x) = \{w \,|\, h(w) = x\}$
and similarly
$f^{-1}(x) = \{w \,|\, f(w) = x\}$.

Using the theorem, see (iv) and (v), we can express $\oplus$ as:

$$(x \oplus y) = h^{-1}(h(x) \oplus h(y)) \cap f^{-1}(f(x) \oplus f(y)).$$

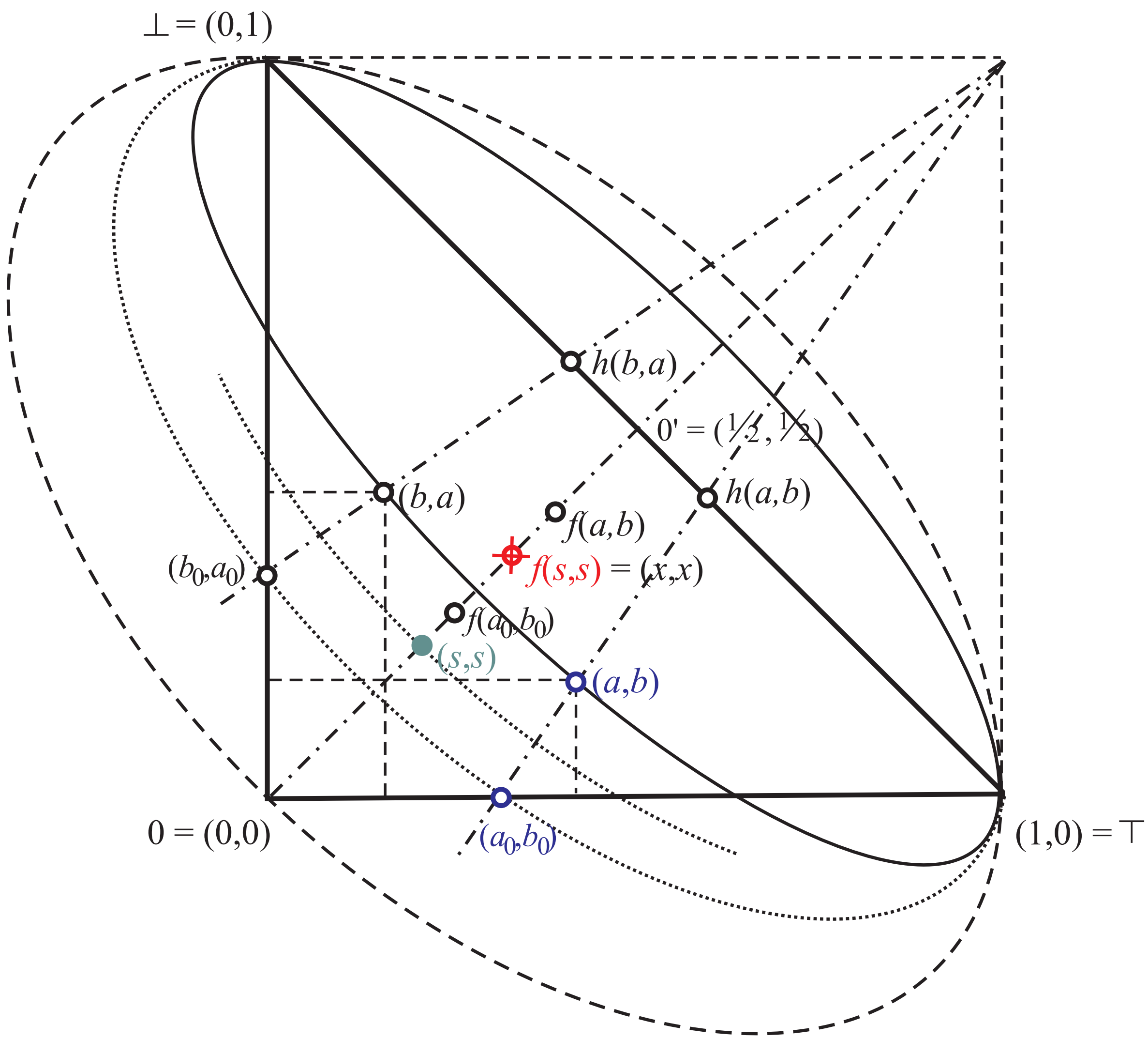
BFs on n-Element Frames of Discernment

We can represent a BF on any n-element frame Ω_n by an enumeration of its m values (bbms), i.e., by a (2^n-2)-tuple $(a_1, a_2, ..., a_{2^n-2})$, or as a (2^n-1)-tuple $(a_1, a_2, ..., a_{2^n-2}; a_{2^n-1})$ when we want to explicitly mention also the redundant value $m(\Omega) = a_{2^n-1} = 1 - \sum_{i=1}^{2^n-2} a_i$.

Unfortunately, no algebraic analysis of BFs on Ω_n for $n > 2$ was presented till now.

Non-conflicting and conflicting parts of BFs on Ω_2

$$(a,b) \quad \oplus \quad (b,a) \quad = \quad f(a,b)$$
$$(a_0,b_0)\oplus(s,s) \quad \oplus \quad (b_0,a_0)\oplus(s,s) \quad = \quad f(a_0,b_0)\oplus f(s,s)$$



$$(a,b) = (a_0,b_0)\oplus(s,s)$$
$$f(a,b) = f(a_0,b_0)\oplus f(s,s)$$

$f(a,b), f(a_0,b_0)$:

$\Rightarrow \quad f(s,s)$

$\Rightarrow \quad (s,s)$

Idea of conflicting and non-conflicting parts

Non-conflicting and conflicting parts of BFs on Ω_2 (cont.)

Proposition 2: Any belief function $(a,b) \in \Omega_2$ is the result of Dempster's combination of BF $(a_0,b_0) \in S_1 \cup S_2$ and a BF $(s,s) \in S$, such that (a_0,b_0) has the same plausibility support as (a,b) does, and (s,s) does not prefer any of the elements of Ω_2. (Trivially, $(s,s) = (0,0) \oplus (s,s)$ for $(s,s) \in S$, and $(a_0,b_0) = (a_0,b_0) \oplus (0,0)$ for elements of S_1, S_2).

$(a_0,b_0) \in S_1 \cup S_2$ ... no internal conflict ... *non-conflicting part*. There is $(a_0,b_0) = (\frac{a-b}{1-b}, 0)$ for $a \geq b$ and $(a_0,b_0) = (0, \frac{b-a}{1-a})$ for $a \leq b$.

Lemma 1: (i) For any BFs (u,u), (v,v) on S, such that $u \leq v$, we can compute their **Dempster's 'difference'** (x,x) such that

$(u,u) \oplus (x,x) = (v,v)$, where $(x,x) = (\frac{v-u}{1-3u+uv}, \frac{v-u}{1-3u+uv})$.

(ii) For any BF (w,w) on S, we can compute its **Dempster's 'half'** (s,s) such that

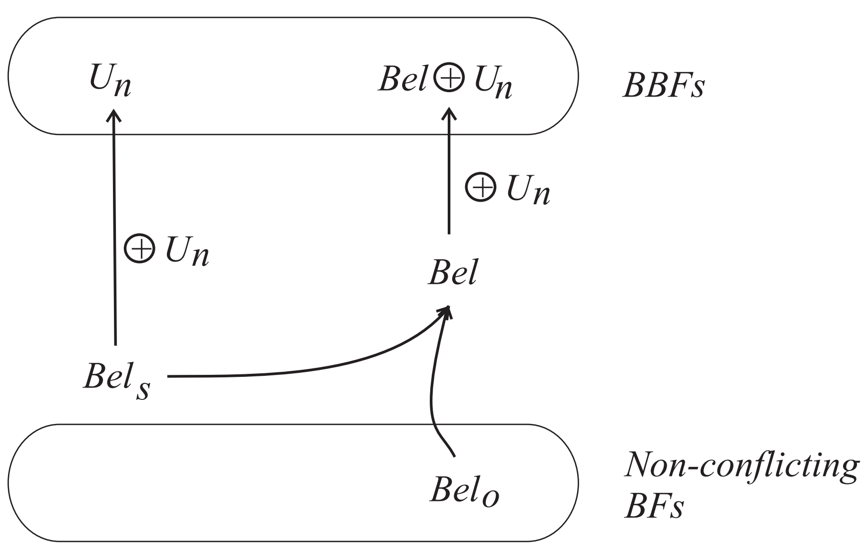
$(s,s) \oplus (s,s) = (w,w)$, where $(s,s) = (\frac{1-\sqrt{1-3w+2w^2}}{3-2w}, \frac{1-\sqrt{(1-w)(1-2w)}}{3-2w})$.

(iii) **There is no Dempster's 'difference' on D_0 in general.**

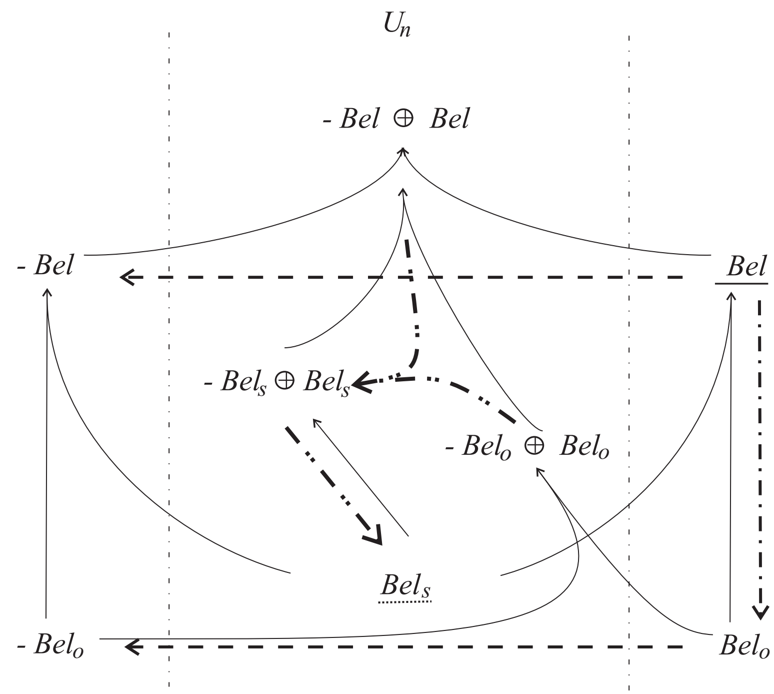
Theorem 2: Any BF (a,b) on Ω_2 is Dempster's sum of its *unique non-conflicting part* $(a_0,b_0) \in S_1 \cup S_2$ and of its *unique conflicting part* $(s,s) \in S$, which does not prefer any element of Ω_2, i.e. $(a,b) = (a_0,b_0) \oplus (s,s)$. It holds true that $s = \frac{b(1-a)}{1-2a+b-ab+a^2} = \frac{b(1-b)}{1-a+ab-b^2}$ and $(a,b) = (\frac{a-b}{1-b}, 0) \oplus (s,s)$ for $a \geq b$ and analogously for $a \leq b$.

Non-conflicting part of BFs on general finite frame Ω_n

Hypothesis 1: We can represent any BF Bel on n-element frame of discernment Ω_n as Dempster's sum $Bel = Bel_0 \oplus Bel_S$ of non-conflicting BF Bel_0 and of indecisive conflicting BF Bel_S which has no decisional support, i.e. which does not prefer any element of Ω_n to the others.



Schema of Hypothesis 1.



Schema of decomposition of a BF

We would like to follow the idea from the case of two-element frames.
Unfortunately, there was not presented any algebraic description of BFs defined on n-element frames till now.

Non-conflicting part of BFs on general frame Ω_n (cont.)

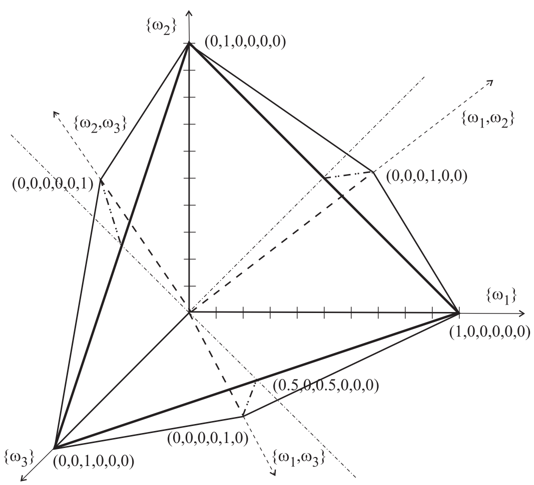
An issue of homomorphism h is quite promising:

Theorem 3: The mapping $h(Bel) = Bel \oplus U_n = Pl\_P(Bel)$ is an **homomorphism** of an algebra of BFs on an n-element frame of discernment with the binary operation of Dempster's sum $\oplus$ and two nulary operations (constants) 0 and U_n to the algebra of BBFs on Ω_n with binary operation $\oplus$ and nulary operation U_n.

Idea of procedure for computing unique consonant BF Bel_0 to any $h(Bel)$:
$h(Bel) = (h_1, h_2, ..., h_n, 0, 0, ..., 0)$; k different values of $h(Bel)(\omega_i) = h_i(Bel)$
disjoint splitting of Ω : $\Omega = \Omega_1 \cup \Omega_2 \cup ... \cup \Omega_k \quad (k \leq n)$
$h(Bel)(\omega_i) = const.$ for $\omega_i \in \Omega_r$ and $h(Bel)(\omega_i) > h(Bel)(\omega_j)$ for $\omega_i \in \Omega_r, \omega_j \in \Omega_s, r > s$
$m_w(\Omega_i) = h(Bel)(\omega_r) - h(Bel)(\omega_s)$, where $\omega_r \in \Omega_i, \omega_s \in \Omega_{i+1}$, $m_w(\Omega_k) = h(Bel)(\omega_j)$, where $\omega_j \in \Omega_k$, $m_w(X) = 0$ otherwise,
Bel_0: m_0 is normalization of m_w.

A simplification using $h(Bel) = Pl\_P(Bel)$ instead of $h(Bel) = Bel \oplus U_n$.
(it removes Dempster's rule hidden in original definition of h)
Any Bel has defined its non-conflicting part Bel_0 independently of any belief combination rule.

Non-conflicting part of BFs on general frame Ω_n (cont.)

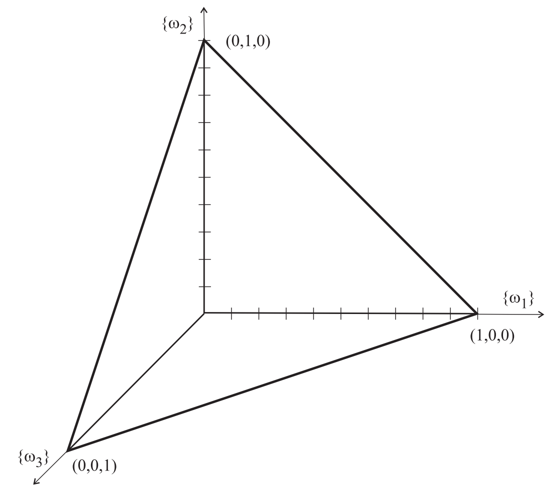


General BF on 3-element frame Ω_3.

Looking for $-Bel$:

idea of complements $(\Omega \setminus X)$ ... does not work in general

simplification to qBBFs ... Bel_0 is frequently outside of 'triangle'



Quasi Bayesian BFs on 3-el. Ω_3.

BBFs:

Lemma 3: For any BBF $(a_1, a_2, ..., a_n, 0, 0, ..., 0; 0)$ such that, $a_i > 0$ for $i = 1, ..., n$, there exists uniquely defined $-(a_1, a_2, ..., a_n, 0, 0, ..., 0; 0) = (x_1, x_2, ..., x_n, 0, 0, ..., 0; 0) = (1/(1 + \sum_{i=2}^{n} \frac{a_1}{a_i}), \frac{a_1}{a_2} x_1, \frac{a_1}{a_3} x_1, ..., \frac{a_1}{a_n} x_1, 0, 0, ..., 0; 0)$ such that,

$$(a_1, a_2, ..., a_n, 0, 0, ..., 0) \oplus -(a_1, a_2, ..., a_n, 0, 0, ..., 0) = U_n.$$

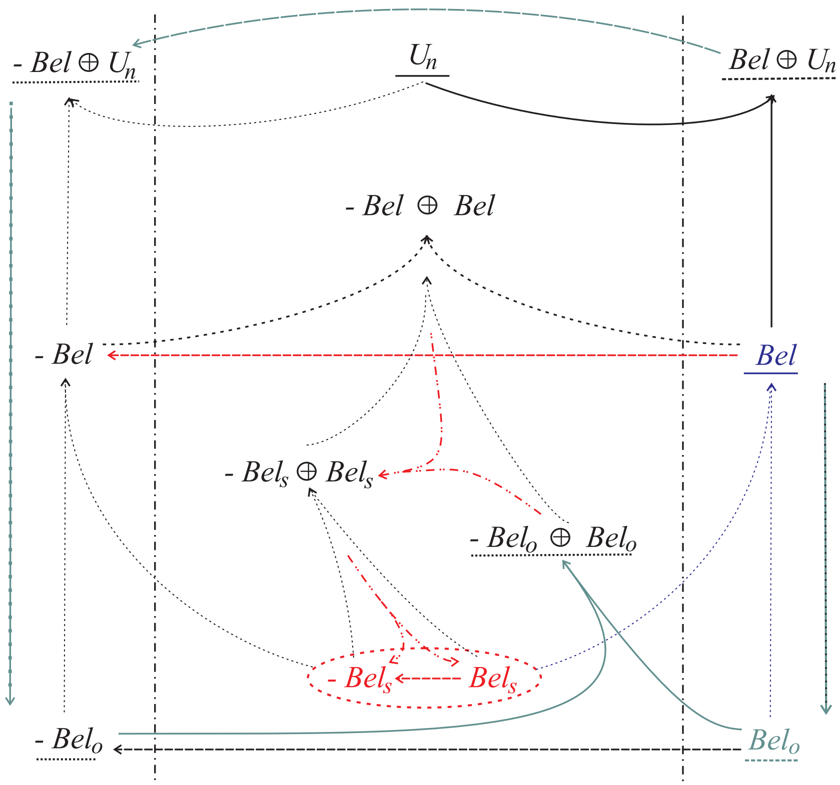
(no $-Bel$ for general BFs, neither for all BBFs; there are still open problems there)

Non-conflicting part of BFs on general frame Ω_n (cont.)

Theorem 4: For any BF Bel defined on Ω_n there exists unique consonant BF Bel_0 such that,

$$h(Bel_0 \oplus Bel_S) = h(Bel)$$

for any BF Bel_S such that $Bel_S \oplus U_n = U_n$.



Schema of current state of decomposition of BF Bel.

If for $h(Bel) = (h_1, h_2, ..., h_n, 0, 0, ..., 0)$ holds that, $0 < h_i < 1$, then further exists unique BF $-Bel_0$ such that, $h(Bel_0) \oplus -h(Bel_0) = U_n$ and $h(-Bel_0 \oplus Bel_S) = -h(Bel)$.

Corollary 1 *(i) For any consonant BF Bel such that $Pl(\{\omega_i\}) > 0$ there exist a unique BF $-Bel$; $-Bel$ is consonant in this case.*

(ii) There is one-to-one correspondence between Bayesian BFs and consonant BFs.

Comments on other rules and probilistic transformations

Other combination rules

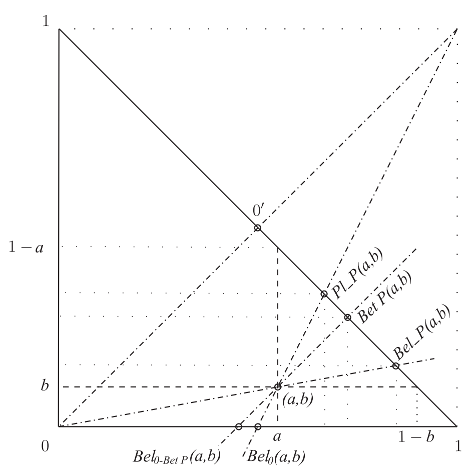
Bel_0 and $Pl\_P(Bel_0) = Pl\_P(Bel)$ independently from any comb. rule.

$Pl\_P(Bel) \neq Bel_0 \textcircled{Y} U_n$, $Pl\_P(Bel) \neq Bel_0 \textcircled{DP} U_n$, $Pl\_P(Bel) \neq Bel_0 \textcircled{\Cup} U_n$

Even $Pl\_P(Bel) \neq Pl\_P(Bel_0 \odot U_n)$, where $\odot$ is either $\textcircled{Y}$, $\textcircled{DP}$, $\textcircled{\Cup}$ or ...

If there exists an analogous couple of homomortphisms for any other rule then ...

Other probabilistic transformations



Probabilistic transformations.

Considering Smets' pignistic pignistic probability $BetP$ we obtain non-conflicting BF $Bel_{0\text{–}BetP}$, where $m_{w\text{–}BetP}(\bigcup_{i=1}^{m} \Omega_i) = |\bigcup_{i=1}^{m} \Omega_i|(h(Bel)(\omega_{m1}) - h(Bel)(\omega_{(m+1)1}))$, which is normalized, hence $m_{w\text{–}BetP} = m_{0\text{–}BetP}$. $BetT$ does not commute with $\oplus$ nor with other ..., thus we cannot use $Bel_{0\text{–}BetP}$ for decomposition.

Bel_P compatible with $\textcircled{\Cup}$ ... but reverse ... $Bel \longmapsto 0$

no similar decomposition of BFs for $\textcircled{Y}$, $\textcircled{DP}$, $\textcircled{\Cup}$ and ...

Ideas for future research

- Algebraic analysis of BFs on a 3-element frame Ω_3.
- Algebraic analysis of BFs on a general finite frame Ω_n.
- Existence and uniqueness of a conflitcting part of BF on a general finite frame Ω_n.
- Interpretation of (s, s) on Ω_2 and of a conflicting part of a BF on a general finite frame Ω_n.

Current related research

F. Cuzzolin — Consistent transformations of BFs. ECSQARU 2011
On consistent approximations of belief functions in the mass space.

F. Cuzzolin — Consonant transformations of BFs. ISIPTA 2011
Lp consonant approximation of belief functions in the mass space.

Lefevre-Elouedi-Mercier — Partial normalization of conflicting mass $m(\emptyset)$ in TBM. ECSQARU 2011 Towards an alarm for opposition conflict in a conjunctive combination of belief functions.

Conclusion

- Decomposition of a belief function (BF) defined on a two-element frame of discernment to Dempster's sum of its unique non-conflicting and unique indecisive conflicting part is defined and presented here.
- Homomorphic properties of mapping $h(Bel) = Bel \oplus U_n$ which corresponds to normalized plausibility of singletons were verified for BFs defined on a general finite frame of discernment.
  $-Bel$ was generalized to Bayesian BFs and for consonant BFs on a general n-element frame, s.t. $Pl(\{\omega_i\}) > 0$ for all $i \leq n$.
- Unique consonant non-conflicting part Bel_0 of a general BF Bel on a finite frame was defined. For specification of a corresponding conflicting part of Bel and its uniqueness/existence properties, **an algebraic analysis of BFs on a general finite frame of discernment is required.**
- Discussion of the topic from the point of view of alternative rules of combination and alternative probabilistic transformations.
- Improvement of gen. understanding of BFs and their combination, especially in conflicting cases.
  One of corner-stones to further study of conflicts between BFs.

THANK YOU FOR YOUR ATTENTION.