Robustness of Natural Extension

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  - Gent $\rightarrow$ Pittsburgh $\rightarrow$ Durham
  - decision making under severe uncertainty
  - computational tools
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  - decision making under severe uncertainty
  - nonparametric statistics
  - robust statistics
  - interval probability
  - earlier work on robustness of natural extension ([1, pp. 118–125, Sec. 5.2] and [2, Sec. 2])

Aim

how sensitive is the natural extension of an upper prevision against small perturbations in the assessments?

- various forms and representations of natural extension: does this affect stability?
- necessary and sufficient conditions?
- transform instable problems into stable ones?

Example 1: Instability of Avoiding Sure Loss

- $\Omega = \{\omega_1, \omega_2\}$, $\mathcal{L} = \mathbb{R}^\omega$, $X = \mathcal{L}^*$
- $\overline{P}(I_{\omega_2}) = 2/3$, $\overline{P}(I_{\omega_1}) = 1/3$
- credal set = subset of X

$$x_1 \geq 0,\ x_2 \geq 0,\ \sum_{i=1}^{2} x_i = 1 \tag{C}$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq \begin{bmatrix} 2/3 \\ 1/3 \end{bmatrix} \tag{S}$$

- natural extension = maximize linear map x over credal set

$$\overline{E}(I_{\omega_1} + 2I_{\omega_2}) \to \text{maximize } \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Example 1: Instability of Avoiding Sure Loss

- small perturbation in assessments yields instability
  - (C) + (S) has a non-empty feasible set
  - but (C) + $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq \begin{bmatrix} 2/3 - \epsilon \\ 1/3 \end{bmatrix}$
    has an empty feasible set, for any $\epsilon > 0$.
- linear programming algorithms will often fail to solve even this simple problem due to simple rounding errors!

Example 1: Stable representation

- the modified system $\begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2/3 \end{bmatrix}$
  has the same feasible region as original problem
- perturbations $\begin{bmatrix} 0 \pm \epsilon & 1 \pm \delta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2/3 \pm \eta \end{bmatrix}$
  have a solution for any ϵ, δ, η sufficiently small
- solution of perturbated problem close to original solution

$$x_2 = 2/3 \sim \hat{x}_2 = \frac{2/3 + \eta - \epsilon}{1 + \delta - \epsilon}$$

- take-home message: recognize implicit linearities

Example 2: Instability of Natural Extension

- adapted from Robinson [3, p. 443]
- $\Omega = \{a, b, c, d\}$

$$\overline{P}(I_a + 2I_b/3 + 2I_d) = 1/2 \qquad \overline{P}(I_b + 3I_c) = 3/2$$

- $\overline{E}$ of $2I_b + 2I_c$:

$$\text{maximize} \quad \begin{bmatrix} 0 & 2 & 2 & 0 \end{bmatrix} \begin{bmatrix} x_a \\ x_b \\ x_c \\ x_d \end{bmatrix}$$

subject to

$$x_a \geq 0,\ x_b \geq 0,\ x_c \geq 0,\ x_d \geq 0 \qquad \text{(C)}$$
$$x_a + x_b + x_c + x_d = 1$$

$$\begin{bmatrix} 1 & 2/3 & 0 & 2 \\ 0 & 1 & 3 & 0 \end{bmatrix} \begin{bmatrix} x_a \\ x_b \\ x_c \\ x_d \end{bmatrix} \leq \begin{bmatrix} 1/2 \\ 3/2 \end{bmatrix} \qquad \text{(S)}$$

Example 2: Instability of Natural Extension

- (C) + (S) have a non-empty feasible set

$$\alpha \begin{bmatrix} 1/2 \\ 0 \\ 1/2 \\ 0 \end{bmatrix} + (1-\alpha) \begin{bmatrix} 0 \\ 3/4 \\ 1/4 \\ 0 \end{bmatrix}$$

- so $\overline{E}(2I_b + 2I_c) = 2$

Example 2: Instability of Natural Extension

- (C) + (S_ϵ), with

$$\begin{bmatrix} 1 & 2/3-\epsilon & 0 & 2 \\ 0 & 1 & 3 & 0 \end{bmatrix} \begin{bmatrix} x_a \\ x_b \\ x_c \\ x_d \end{bmatrix} \leq \begin{bmatrix} 1/2 \\ 3/2 \end{bmatrix} \tag{S_ϵ}$$

  has only one feasible solution

$$\begin{bmatrix} 1/2 \\ 0 \\ 1/2 \\ 0 \end{bmatrix}$$

- so, now, $\overline{E}(2I_b + 2I_c) = 1$
- take-home message:
  arbitrary small perturbation in assessments can have disproportionally large effect on natural extension

Example 2: Bewildering Facts

- instability in example remains
  even after recognizing implicit linearities
- stable representation?
- example has perturbations that incur sure loss
- dual problem has an unbounded optimal solution
- example has arbitrarily close stable approximations:

$$(1-\alpha)\overline{P} + \alpha \operatorname{sup}$$

is stable for *any* $0 < \alpha \leq 1$

all of the above are expressions of
the same mathematical result

Do you want to know why?
Come to the poster!

Thank you!!

References

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