Modelling Uncertainty in Limit State Functions

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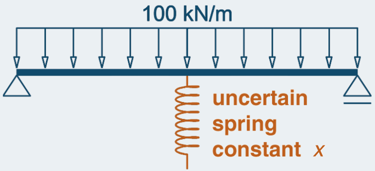
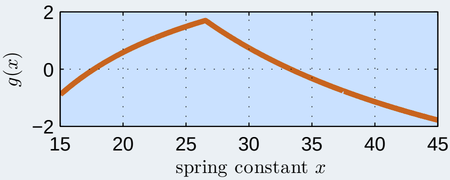
Limit state functions

- $g : \mathcal{X} \subseteq \mathbb{R}^n \rightarrow \mathcal{Y} \subseteq \mathbb{R} : \; x \rightarrow y = g(x)$
- $x = (x_1, \ldots, x_n) =$ material properties, loads,... (basic variables)
- $y = g(x) \leq 0$ means failure of the system.

Beam bedded on a spring with uncertain spring constant x





Limit state functions / probability of failure

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Failure regions

- $R_f = \{x \in \mathcal{X} : \; g(x) \leq 0\}$ described by $q : \mathcal{X} \to \{0, 1\} : x \to \chi(g(x) \leq 0)$.

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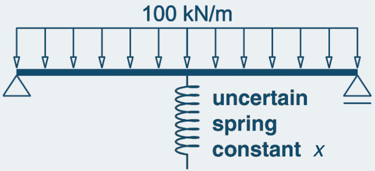

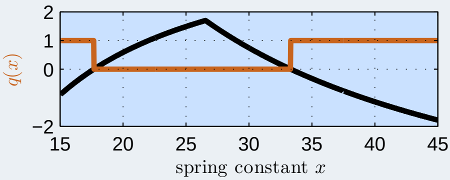

Limit state functions / probability of failure

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Probability of failure

- The values of $x = (x_1, \ldots, x_n)$ are assumed to be uncertain.
- f^X density function of the random variables $X = (X_1, \ldots, X_n)$.

$$p_f = P(g(X) \leq 0) = \int_{\mathcal{X}} \chi(g(x) \leq 0) f^X(x) \, \mathrm{d}x = \int_{\mathcal{X}} q(x) f^X(x) \, \mathrm{d}x$$

Probability of failure $p_f(a,b)$ depending on a and b where

- the parameters a are used to model the uncertainty of the basic variables x of a limit state function.
- the parameters b are used to model the uncertainty in the limit state function itself.
- the parameters a and b assumed to be uncertain which is described by sets or random sets
  $\rightarrow$ **sets of probability measures**
  $\rightarrow$ **upper probabilities of failure $\overline{p}_f$.**

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Outline of the presentation

- Step 1: Parameterized limit state functions $\rightarrow$ independence.
- Step 2: Introducing $p_f(a,b)$.
- Step 3: Generating sets of probability measures.
- Sets of probability measures and notion independence.
- Step 4: Computational formulas for the upper probability of failure.

Step 1: Parameterized limit state functions

Parameterization g_z of a limit state function g

- $h : \mathcal{X} \times \mathcal{Z} \to \mathcal{Y} : (x,z) \to h(x,z) = g_z(x), \quad z \in \mathcal{Z} \subseteq \mathbb{R}^m$,
- $y = h(x,z) \leq 0$ means failure, values of z are uncertain.

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Probability of failure

$$p_f = \int_{\mathcal{X}} \int_{\mathcal{Z}} \chi(h(x,z) \leq 0) f^Z(z)\, \mathrm{d}z\, f^X(x)\, \mathrm{d}x$$

assuming independence of X and Z, f^Z density of Z.

Independence of X and Z

(a) *If we learn the values of the variables x, our knowledge about the parameters z and therefore about the choice of the limit state functions g_z does not change.*

(b) *Learning the values of z and therefore learning which function g_z to use has no influence on our knowledge about the variables x.*

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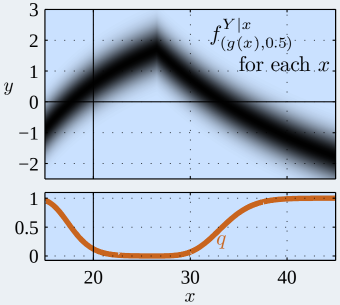
Imprecise failure regions

$$q : \mathcal{X} \to [0,1] : x \to \int_{\mathcal{Z}} \chi(h(x,z) \leq 0) f^Z(z)\, \mathrm{d}z,$$

cf. membership function of a fuzzy set.

Random limit state function

$Y_{|x} = h(x,Z) = g(x) + Z$,
$Z \sim \mathcal{N}(\mu, \sigma^2)$, $Y_{|x} \sim \mathcal{N}(g(x) + \mu, \sigma^2)$



Probability of $p_f(a,b)$

$$p_f(a,b) = \int_{\mathcal{X}} \int_{\mathcal{Z}} \chi(h(x,z) \leq 0)\, f_b^Z(z)\, \mathrm{d}z\, f_a^X(x)\, \mathrm{d}x$$

- Arguments a and b are parameters of the densities f_a^X and f_b^Z, e. g. $f_{(\mu,\sigma)}^X$ density of a Gaussian distribution $\mathcal{N}(\mu,\sigma^2)$.
- Parameters a and b are assumed to be uncertain described by sets or random sets
  $\rightarrow$ **sets of probability measures $\mathcal{M}_X$ for x and $\mathcal{M}_Z$ for z**
  $\rightarrow$ **upper probabilities of failure $\overline{p}_f$**.
- Different notions of independence!
- To model the uncertainty of x and z directly by sets or random sets: Replace the density functions by Dirac measures.

Step 3: Generating sets $\mathcal{M}_X$ and $\mathcal{M}_Z$ of probability measures

The unceratianties of x and z are described by $\mathcal{M}_X$ and $\mathcal{M}_Z$.

Uncertainty of parameters a and b modelled by sets A and B

$$\mathcal{M}_X = \left\{ P\colon P(E) = \int_{\mathcal{A}} \int_{\mathcal{X}} \chi(x \in E) f_a^X(x)\, \mathrm{d}x\ \mathrm{d}P_A(a),\ P_A \in \mathcal{M}(A) \right\}$$

$\mathcal{M}(A) = \{P :\ P(A) = 1\}$ is the set of all probability measures living on the set $A \subseteq \mathcal{A}$.

$\mathcal{M}_Z$ is obtained in a similar way using f_b^Z and B.

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Uncertainty of a and b modelled by finite random sets $\mathscr{A}$ and $\mathscr{B}$

Random sets $\mathscr{A}$, $\mathscr{B}$ with focal sets A_i, B_j and weights $m_{\mathscr{A}}(A_i)$, $m_{\mathscr{B}}(B_j)$.

$$\mathcal{M}_X = \Big\{P:\ P(E) = \sum_{i=1}^{|\mathscr{A}|} m_{\mathscr{A}}(A_i) \int_{\mathcal{A}}\int_{\mathcal{X}} \chi(x\in E)\, f_a^X(x)\,\mathrm{d}x\;\mathrm{d}P_{A_i}(a),\; P_{A_i} \in \mathcal{M}(A_i)\Big\}$$

with $\mathcal{M}(A_i) = \{P:\ P(A_i) = 1\}$. $\mathcal{M}_Z$ is obtained using f_b^Z and $\mathscr{B}$.

Upper probability of failure and independence

Strong independence

All possible product measures $P_X \otimes P_Z$ for $P_X \in \mathcal{M}_X$ and $P_Z \in \mathcal{M}_Z$:

$$\overline{p}_f^S = \sup\left\{(P_X \otimes P_Z)(h(x,z) \leq 0) : P_X \in \mathcal{M}_X, P_Z \in \mathcal{M}_Z\right\}$$

$$= \sup_{\substack{P_X \in \mathcal{M}_X \\ P_Z \in \mathcal{M}_Z}} \int_{\mathcal{X}} \int_{\mathcal{Z}} \chi(h(x,z) \leq 0) \, \mathrm{d}P_Z(z) \, \mathrm{d}P_X(x)$$

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Set of imprecise failure regions

$$Q = \left\{ q : q(x) = \int_{\mathcal{Z}} \chi(h(x,z) \leq 0)\, \mathrm{d}P_Z(z),\, P_Z \in \mathcal{M}_Z \right\}$$

- The entire set Q is needed for computations!

Strong independence, swap supremum and integral

$$\overline{p}_f^S = \sup_{P_X \in \mathcal{M}_X} \sup_{P_Z \in \mathcal{M}_Z} \int_{\mathcal{X}} \int_{\mathcal{Z}} \chi(h(x,z) \leq 0) \, \mathrm{d}P_Z(z) \, \mathrm{d}P_X(x)$$

$$= \sup_{P_X \in \mathcal{M}_X} \sup_{q \in Q} \int_{\mathcal{X}} q(x) \, \mathrm{d}P_X(x)$$

Upper probability of failure / independence

Strong independence → Epistemic irrelevance

$$\overline{p}_f^{\mathrm{S}} \leq \sup_{P_X \in \mathcal{M}_X} \int_{\mathcal{X}} \sup_{P_Z \in \mathcal{M}_Z} \int_{\mathcal{Z}} \chi(h(x,z) \leq 0)\, \mathrm{d}P_Z(z)\, \mathrm{d}P_X(x)$$

$$= \sup_{P_X \in \mathcal{M}_X} \int_{\mathcal{X}} \sup_{q \in Q} q(x)\, \mathrm{d}P_X(x) = \sup_{P_X \in \mathcal{M}_X} \int_{\mathcal{X}} \overline{q}(x)\, \mathrm{d}P_X(x) =: \overline{p}_f^{X \not\to Z}$$

Strong independence → Epistemic irrelevance

$$\overline{p}_f^{\mathrm{S}} \leq \sup_{P_X \in \mathcal{M}_X} \int_{\mathcal{X}} \sup_{P_Z \in \mathcal{M}_Z} \int_{\mathcal{Z}} \chi(h(x,z) \leq 0)\, \mathrm{d}P_Z(z)\, \mathrm{d}P_X(x)$$

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Epistemic independence

- Each x can choose its own $P_Z \in \mathcal{M}_Z$ or more exactly a $P_Z(\cdot|x)$ given x.
- $X \not\rightarrow Z$ means that X is epistemically irrelevant to Z or that the basic variables x are epistemically irrelevant to the parameterized limit state functions g_z.
- Epistemic irrelevance is an asymmetric notion of independence meaning only part (a): *If we learn the values of the variables x, our knowledge about the parameters z and therefore about the choice of the limit state functions g_z does not change.*

Strong independence $\to$ Epistemic irrelevance

$$\overline{p}_f^{S} \leq \sup_{P_X \in \mathcal{M}_X} \int_{\mathcal{X}} \sup_{P_Z \in \mathcal{M}_Z} \int_{\mathcal{Z}} \chi(h(x,z) \leq 0)\, \mathrm{d}P_Z(z)\, \mathrm{d}P_X(x)$$

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Upper envelope of the failure regions

$$\overline{q}(x) = \sup_{P_Z \in \mathcal{M}_Z} \int_{\mathcal{Z}} \chi(h(x,z) \leq 0)\, \mathrm{d}P_Z(z) = \sup_{q \in Q} q(x).$$

- It is sufficient to specify the function $\overline{q}$!

Uncertainties modelled by ordinary sets A and B

- Strong independence:

$$\overline{p}_f^S = \sup_{\substack{a\in A\\ b\in B}} p_f(a,b)$$

- Epistemic irrelevance:

$$\overline{p}_f^{X\not\to Z} = \sup_{a\in A} \int_{\mathcal{X}} \overline{q}(x) f_a^X(x)\, \mathrm{d}x$$

$$\overline{q}(x) = \sup_{b\in B} \int_{\mathcal{Z}} \chi(h(x,z)\le 0) f_b^Z(z)\, \mathrm{d}z$$

Step 3: Formulas for the upper probability of failure

Uncertainties modelled by random sets $\mathscr{A}$ and $\mathscr{B}$

- Strong independence:

$$\overline{p}_f^S = \sup_{\substack{a_r \in A_r,\, r=1,\ldots,|\mathscr{A}| \\ b_s \in B_s,\, s=1,\ldots,|\mathscr{B}|}} \sum_{i=1}^{|\mathscr{A}|} \sum_{j=1}^{|\mathscr{B}|} m_{\mathscr{A}}(A_i) m_{\mathscr{B}}(B_j)\, p_f(a_i, b_j)$$

- Epistemic irrelevance:

$$\overline{p}_f^{X \not\to Z} = \sum_{i=1}^{|\mathscr{A}|} m_{\mathscr{A}}(A_i) \sup_{a \in A_i} \int_{\mathcal{X}} \overline{q}(x)\, f_a^X(x)\, \mathrm{d}x$$

$$\overline{q}(x) = \sum_{j=1}^{|\mathscr{B}|} m_{\mathscr{B}}(B_j) \sup_{b \in B_j} \int_{\mathcal{Z}} \chi(h(x,z) \le 0)\, f_b^Z(z)\, \mathrm{d}z$$

Step 3: Formulas for the upper probability of failure

Random set independence:

$$\overline{p}_f^{\mathrm{RS}} = \sum_{i=1}^{|\mathscr{A}|}\sum_{j=1}^{|\mathscr{B}|} m_{\mathscr{A}}(A_i)m_{\mathscr{B}}(B_j) \sup_{\substack{a\in A_i\\ b\in B_j}} p_f(a,b),$$ cf. $\overline{p}_f^{\mathrm{S}} = \sup_{\substack{a\in A\\ b\in B}} p_f(a,b)$ for sets A, B.

$$\overline{p}_f^{\mathrm{R},X\not\leftrightarrow Z} = \sum_{i=1}^{|\mathscr{A}|}\sum_{j=1}^{|\mathscr{B}|} m_{\mathscr{A}}(A_i)m_{\mathscr{B}}(B_j) \sup_{a\in A_i} \int_{\mathcal{X}} \overline{q}_j(x)\, f_a^X(x)\,\mathrm{d}x,$$ cf. $\overline{p}_f^{X\not\leftrightarrow Z} = \sup_{a\in A} \int_{\mathcal{X}} \overline{q}(x)\, f_a^X(x)\,\mathrm{d}x$

$$\overline{q}_j(x) = \sup_{b\in B_j} \int_{\mathcal{Z}} \chi(h(x,z)\le 0)\, f_b^Z(z)\,\mathrm{d}z\,.$$

Random set independence:

$$\overline{p}_f^{\mathrm{RS}} = \sum_{i=1}^{|\mathscr{A}|}\sum_{j=1}^{|\mathscr{B}|} m_{\mathscr{A}}(A_i)m_{\mathscr{B}}(B_j) \sup_{\substack{a\in A_i\\ b\in B_j}} p_f(a,b), \text{ cf. } \overline{p}_f^{\mathrm{S}} = \sup_{\substack{a\in A\\ b\in B}} p_f(a,b) \text{ for sets } A, B.$$

$$\overline{p}_f^{\mathrm{R},X\not\perp Z} = \sum_{i=1}^{|\mathscr{A}|}\sum_{j=1}^{|\mathscr{B}|} m_{\mathscr{A}}(A_i)m_{\mathscr{B}}(B_j) \sup_{a\in A_i} \int_{\mathcal{X}} \overline{q}_j(x) f_a^X(x)\,\mathrm{d}x, \text{ cf. } \overline{p}_f^{X\not\perp Z} = \sup_{a\in A} \int_{\mathcal{X}} \overline{q}(x) f_a^X(x)\,\mathrm{d}x$$

Random set independence together with density functions

Joint plausibility measure

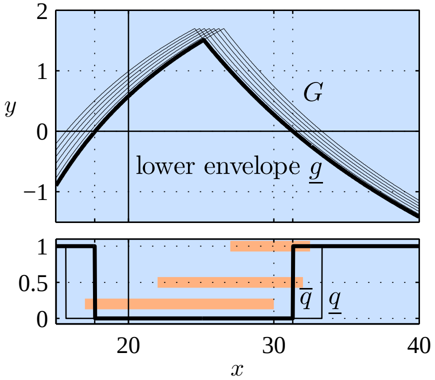
$$\mathrm{Pl}(E) = \sum_{i=1}^{|\mathscr{A}|}\sum_{j=1}^{|\mathscr{B}|} m_{\mathscr{A}}(A_i)m_{\mathscr{B}}(B_j) \overbrace{\chi(E\cap(A_i\times B_j)\neq\emptyset)}^{\sup_{P\in\mathcal{M}(A_i\times B_j)} P(E)}$$

"Correct" generalization of Pl: All possible combinations of f_a^X and f_b^Z.
High computational effort, no independence on the level of f_a^X and f_b^Z.

Here: We replace $\sup_{P\in\mathcal{M}(A_i\times B_j)} P(E)$ by $\overline{p}_f^{\mathrm{S}}$ and $\overline{p}_f^{X\not\perp Z}$ for ordinary sets.

For Dirac measures all these approaches coincide with Pl!

Examples: Sets of parameterized limit state functions



Uncertainty of x:

$f_a^X := \delta_x$, $a := x$,

random set $\mathscr{A}$ given by three focal sets A_i and weights $m_{\mathscr{A}}(A_i)$.

Parameterization:

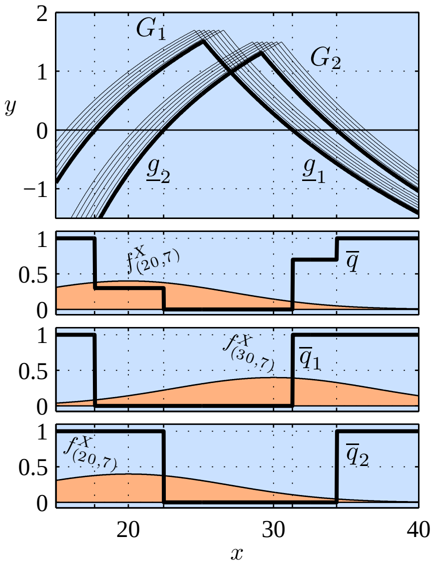
$h(x,z) = g_z(x) = g(x+z)$.

Uncertainty of z:

$f_b^Z := \delta_z$, $b := z$, $z \in B = [\underline{b}, \overline{b}]$,

$G = \{g_z : g_z(x) = h(x,z), z \in B\}$.

Lower envelope $\underline{g}$ corresponds with upper envelope $\overline{q}$.



Uncertainty of x:

f_a^X is the density of a Gaussian distribution with parameters $a = (\mu, \sigma)$ where the parameter a is uncertain and modelled by the set

$$A = \{(\mu, \sigma) : (\mu, \sigma) \in [20, 30] \times \{7\}\}$$

Parameterization:

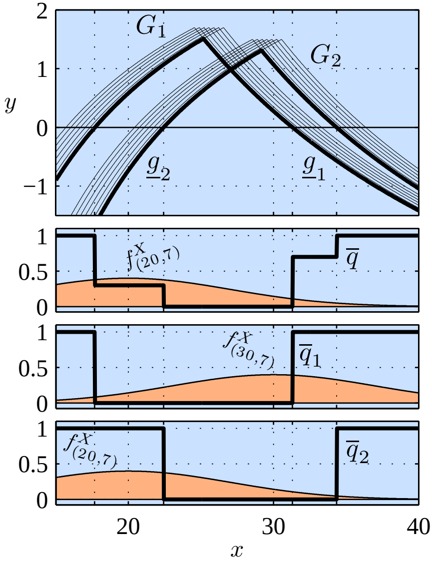
$$h(x, z) = g_z(x) = g(x + z_1) - z_2$$

Uncertainty of z:

$f_b^Z := \delta_z$, $b := z$, random set $\mathscr{B}$ given by two focal sets $B_1 = [0, 2] \times \{0\}$ and $B_2 = [-4, -2] \times \{0.2\}$ with weights $m_{\mathscr{B}}(B_1) = 0.7$ and $m_{\mathscr{B}}(B_2) = 0.3$.

$$G_1 = \left\{g_z : g_z(x) = g(x + z_1) - z_2, z \in B_1\right\}$$
$$G_2 = \left\{g_z : g_z(x) = g(x + z_1) - z_2, z \in B_2\right\}$$

Examples: Random sets of parameterized limit state functions



epistemic irrelevance, $\overline{p}_f^{X \not\to Z}$

separate optimization + weighted sum

= random set independence

$\overline{p}_f^{\mathrm{R},X \not\to Z}$

Summary

- **Orderings of the upper probabilities of failure:**

  $\overline{p}_f^{\mathrm{S}} \leq \overline{p}_f^{X \not\to Z} \leq \overline{p}_f^{\mathrm{R},X \not\to Z}$ and $\overline{p}_f^{\mathrm{S}} \leq \overline{p}_f^{\mathrm{RS}} \leq \overline{p}_f^{\mathrm{R},X \not\to Z}$.

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- **Strong independence:** Complete information about the uncertain limit state function is needed.

- **Epistemic irrelevance:** Sufficient to know the function $\overline{q}$ which condenses the uncertain limit state function.

  $\overline{q}$ could also be a starting point for uncertainty modelling.

- **Random set independence:** Possibility to combine upper probabilities of failure resulting from different and independent computations.

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- **Random set independence:** Possibility to combine upper probabilities of failure resulting from different and independent computations.

- The amount of information to deal with decreases from the uncertain limit state function itself to the function $\overline{q}$ and to single upper probabilities which is reflected in the computational effort and in the above orderings of the upper probabilities.