LINEAR PROGRAMMING UNDER VACUOUS AND POSSIBILISTIC UNCERTAINTY

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I. LINEAR PROGRAMMING UNDER UNCERTAINTY (LPU)

General LPU	A toy problem
$\max_{x\in\mathbb{R}^n} U^T x$	$\max_{(x_1,x_2)\in\mathbb{R}^2} u_1x_1+u_2x_2$
subject to $Yx \le Z$	subject to $\begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22}\end{bmatrix}\begin{bmatrix} x_1 \\ x_2\end{bmatrix} \le \begin{bmatrix} z_1 \\ z_2\end{bmatrix}$
$x \ge 0$	$x_1, x_2 \ge 0$
U, Y and Z are independent random variables taking values u in $\mathbb{R}^n$, y in $\mathbb{R}^{m\times n}$ and z in $\mathbb{R}^m$	$u_1 \in [5,6], y_{21} \in [9,10], y_{22} = [7,8], z_2 = [11,12], u_2 \in [11,12], z_1 \in [5,6], y_{12} = [-4,-3], y_{11} \in [1,2]$

II. PROBLEM STATEMENT

Three types of problems can be derived:

LPU I	LPU II	General LPU
$\max_{x\in\mathbb{R}^n} c^T x$	$\max_{x\in\mathbb{R}^n} U^T x$	$\max_{x\in\mathbb{R}^n} U^T x$
s.t. $Yx \le Z,\ x \ge 0,\ c \in \mathbb{R}^n$	s.t. $Yx \le b,\ x \ge 0,\ b \in \mathbb{R}^m$	s.t. $Yx \le Z,\ x \ge 0$

But by using the *epigraph* form, all of other types reduced to only one type:

General LPU	Non-standard LPU	LPU I
$\max_{x\in\mathbb{R}^n} U^T x$	$\max_{(x,x_0)\in(\mathbb{R}^n,\mathbb{R})} x_0$	$\max_{(x,x_0^+,x_0^-)\in(\mathbb{R}^n,\mathbb{R},\mathbb{R})} x_0^+ - x_0^-$
s.t. $\begin{cases} Yx \le Z \\ x \ge 0 \end{cases}$	s.t. $\begin{cases} Yx \le Z,\ x_0 \le U^T x \\ x \ge 0,\ x_0 \in \mathbb{R} \end{cases}$	$\Rightarrow$ s.t. $\begin{cases} Yx \le Z,\ x_0^+ - x_0^- - U^T x \le 0 \\ x, x_0^+, x_0^- \ge 0 \end{cases}$

III. UNCERTAINTY MODELS

Coherent lower prevision $\underline{P}_V$, $V := (Y,Z,U)$

Vacuous prevision on $\mathscr{A} \subseteq \mathscr{Y} := A \times B \times C$,

$$A := \bigtimes_{k=1}^{m} \bigtimes_{l=1}^{n} A_{kl};\ A_{kl} := [\underline{y}_{kl}, \overline{y}_{kl}],\ B := \bigtimes_{k=1}^{m} B_k;\ B_k := [\underline{z}_k, \overline{z}_k],\ C := \bigtimes_{k=1}^{m} C_k;\ C_k := [\underline{u}_k, \overline{u}_k]$$

If $g = g(V)$ on $\mathscr{A}$, $\underline{P}_V(g) := \min_{v\in\mathscr{A}} g(v)$.

Possibility distribution π: Suppose that the uncertainty about Y, Z, and U is described by π_Y, π_Z, and π_U respectively and that $\pi(v) := \min(\pi_Y(y), \pi_Z(z), \pi_U(u))$, on $A := \operatorname{supp}(\pi)$, where

$$\pi_Y(y) := \min_{1\le k\le m} \min_{1\le l\le n} \pi_{Y_{kl}}(y_{kl}),\ \pi_Z(z) := \min_{1\le k\le m} \pi_{Z_k}(z_k),\ \pi_U(u) := \min_{1\le k\le m} \pi_{U_k}(u_k)$$

If $h = h(V)$ on A, $\overline{P}_V(h) := \int_0^1 \sup\{h(v) : \pi(v) \ge t\}dt$

IV. METHODOLOGY

Reformulate the original problem as a decision problem:

1. For any decision $x \ge 0$ define a gain function $G_x := (U^T x - L)I_{Yx\le Z} + L$, where L is small enough.
2. Use decision criteria that takes into account the uncertainty about V:

maximin solutions	maximal solutions
Pick the x that maximizes $\underline{P}_V(G_x)$	Pick the x for which $\min_{z\ge 0} \overline{P}_V(G_x - G_z) \ge 0$

3. Solve the decision problem:

Model	Maximin solutions	Maximal solutions
Vacuous	Classical LP problem	Classical feasibility problem
P. D.	Classical optimization problem	We have not found an efficient way for calculation, however approximation can be applied when m and n are small enough.

4. Calculation.

V. SOLUTIONS FOR VACUOUS PREVISION MODEL

Maximinity	Maximality
$\operatorname{argmax}_{x\in\mathbb{I}} c^T x$ $\mathbb{I} := \bigcap_{(y,z)\in\mathscr{A}} \{x \ge 0 : yx \le z\}$	$\{x \in \mathbb{R}^n : x \in \mathbb{O} \text{ and } c^T x \ge c^T x_m,\ x \ge 0\}$ where x_m is maximin solution and $\mathbb{O} := \bigcup_{(y,z)\in\mathscr{A}} \{x \ge 0 : yx \le z\}$
$\max_{x\in\mathbb{R}^n} c^T x$ $\overline{y}x \le \underline{z},\ x \ge 0$	$\{x \ge 0 : c^T x \ge c^T x_m \text{ and } \underline{y}x \le \overline{z}\}$ $\underline{y}x \le \overline{z}$ and $\overline{y}x \le \underline{z}$ are the outer and inner feasibility space, respectively

The maximal solutions can be found by vertex enumeration because of the convexity of maximal solutions.

VI. MAXIMIN SOLUTIONS FOR POSSIBILITY DISTRIBUTION MODEL

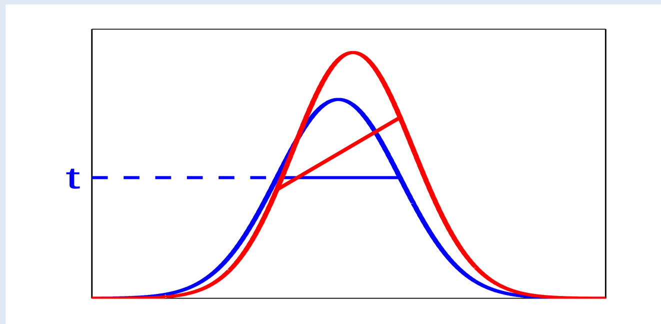
$$\operatorname{argmax}_{x\ge 0} (c^T x - L)\,\underline{P}_V(Yx \le Z)$$

When the possibility distribution is unimodal then the solution is unimodal too because of the linearity of the goal function, then by using the bisection method we can find the maximin solution; in each step a linear programming problem must be solved:

$\operatorname{argmax}_{x\ge 0} [1 - \underline{S}(x)]\,(c^T x - L)$

where $\underline{S}(x) := \min\{t : x \in \mathbb{I}_t\}$,

where $\mathbb{I}_t$ is the inner feasibility space in each step t.



VII. MAXIMIN SOLUTIONS FOR LPU II VIA DUALITY

LPU	Vacuous Prevision	Possibility Distribution
Primal	Dual solution	Exact solution
$\max_{x\in\mathbb{R}^n} U^T x$	$\min_{w\in\mathbb{R}^m} b^T w$	$\operatorname{argmax}_{w\in\mathbb{R}^m} [1 - \underline{S}(w)]\,(b^T w - L)$,
$Yx \le b$	$\overline{y}^T w \le \underline{u},\ w \ge 0$	$\underline{S}(w) := \min_{(t,w)\in(\mathbb{R},\mathbb{R}^m)} t$
$x \ge 0$	(inner feasibility space of dual)	subject to $\overline{y}_t^T w \le \underline{u}_t,\ t \in [0,1]$
Dual	Dual of dual-solution	Approximation
$\min_{w\in\mathbb{R}^m} b^T w$	$\max_{x\in\mathbb{R}^n} \underline{u}^T x$	$\operatorname{argmax}_{w\in\mathbb{R}^m} \left[(b^T w - L)\frac{1}{n}\sum_{k=1}^n I_{\pi_k^*}(w)\right]$,
$Y^T w \le U$	$\overline{y}x \le b,\ x \ge 0$	$I_{\pi_k^*}(w) := \begin{cases} 1, & w \in \pi_k^* \\ 0, & \text{otherwise} \end{cases}$
$U \ge 0$	(inner feasibility space)	π_k^* is inner feasibility space in each level k.

VIII. EXAMPLE I (VACUOUS MODEL)

The LPU is defined as:

$$\max_{(x_1,x_2)\in\mathbb{R}^2} x_1 + x_2 \qquad z_1 \in [5,6], z_2 = [11,12], y_{21} = [9,10]$$

$$\text{subject to } \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22}\end{bmatrix}\begin{bmatrix} x_1 \\ x_2\end{bmatrix} \le \begin{bmatrix} z_1 \\ z_2\end{bmatrix} \qquad y_{11} \in [1,2], y_{12} \in [-4,-3], y_{22} = [7,8]$$

$$x_1, x_2 \ge 0$$

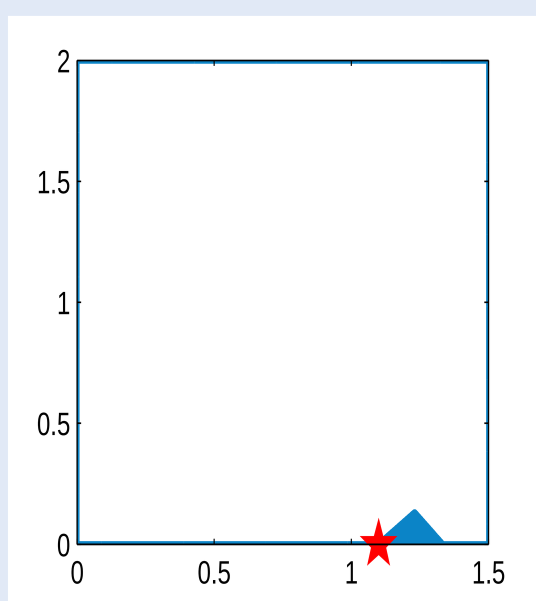
Maximinity solutions	Maximality solutions
$\max_{(x_1,x_2)\in\mathbb{R}^2} x_1 + x_2$ subject to $\begin{cases} 2x_1 - 3x_2 \le 5 \\ 10x_1 + 8x_2 \le 11 \\ x_1, x_2 \ge 0 \end{cases}$	$\begin{cases} x_1 - 4x_2 \le 6 \\ 9x_1 + 7x_2 \le 12 \\ x_1 + x_2 \ge \frac{11}{8},\ x_1, x_2 \ge 0 \end{cases}$

IX. EXAMPLE II (POSSIBILITY DISTRIBUTION)

P.D. example	epigraph form of the toy problem
$\max_{(x_1,x_2)} x_1 - x_2$ $\begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22}\end{bmatrix}\begin{bmatrix} x_1 \\ x_2\end{bmatrix} \le \begin{bmatrix} z_1 \\ z_2\end{bmatrix},\ x_1, x_2 \ge 0$ $z_1 = (0;0.99;1),\ z_2 = (0;0.9;1),\ y_{11} = (0;0.5;1)$ $y_{12} = (0;0.16;0.2),\ y_{22} = (0;0.6;1),\ y_{21} = (0;0.7;1)$	$\max_{(x_1,x_2,x_0^+,x_0^-)} x_0^+ - x_0^-$ $\begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22}\end{bmatrix}\begin{bmatrix} x_1 \\ x_2\end{bmatrix} \le \begin{bmatrix} z_1 \\ z_2\end{bmatrix}$ $x_0^+ - x_0^- - u_1x_1 - u_2x_2 \le 0$ $x_1, x_2, x_0^+, x_0^- \in [0, +\infty)$
For instance in one of the steps when $t = 0.5$ we have $\mathbb{I}_t = \left\{x \ge 0 : \begin{bmatrix} 0.5 & 0.16 \\ 0.7 & 0.6\end{bmatrix}\begin{bmatrix} x_1 \\ x_2\end{bmatrix} \le \begin{bmatrix} 0.99 \\ 0.9\end{bmatrix}\right\}$ after 10 steps we have maximin solution $x_1 = 0$ and $x_2 = 0.9643$	$\max_{(x_1,x_2,x_0^+,x_0^-)} x_0^+ - x_0^-$ $\begin{cases} x_1 - 4x_2 \le 5 \\ 10x_1 + 8x_2 \le 11 \\ x_0^+ - x_0^- - 5x_1 - 11x_2 \le 0 \\ x_1, x_2, x_0^+, x_0^- \ge 0 \end{cases}$

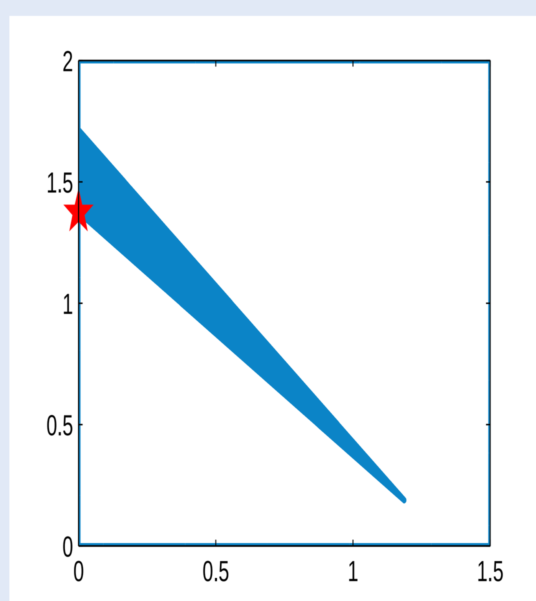
X. GRAPHICAL VIEW OF SOME SOLUTIONS

Maximin solutions for the toy problem



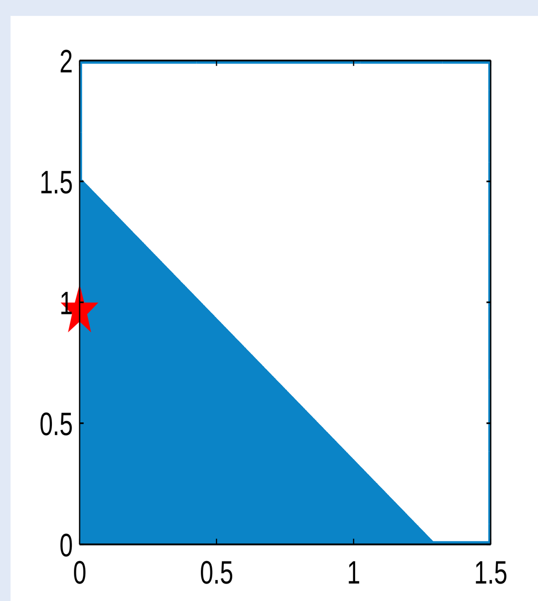
The feasibility space is blue, the red star is the maximin solution

Solutions for example I



The maximal solutions are given in blue, the red star is the maximin solution

Solution for the P.D. example



The feasibility space for $t = 0.5$ is blue, the red star is the maximin solution