TECHNISCHE UNIVERSITÄT BERGAKADEMIE FREIBERG

Die Ressourcenuniversität. Seit 1765.

The Extension Principle Applied to Stochastic Differential Equations

Dipl.-Math. Björn Sprungk [*] and Prof. Dr. rer. nat. K. Gerald van den Boogaart[**]

Motivation

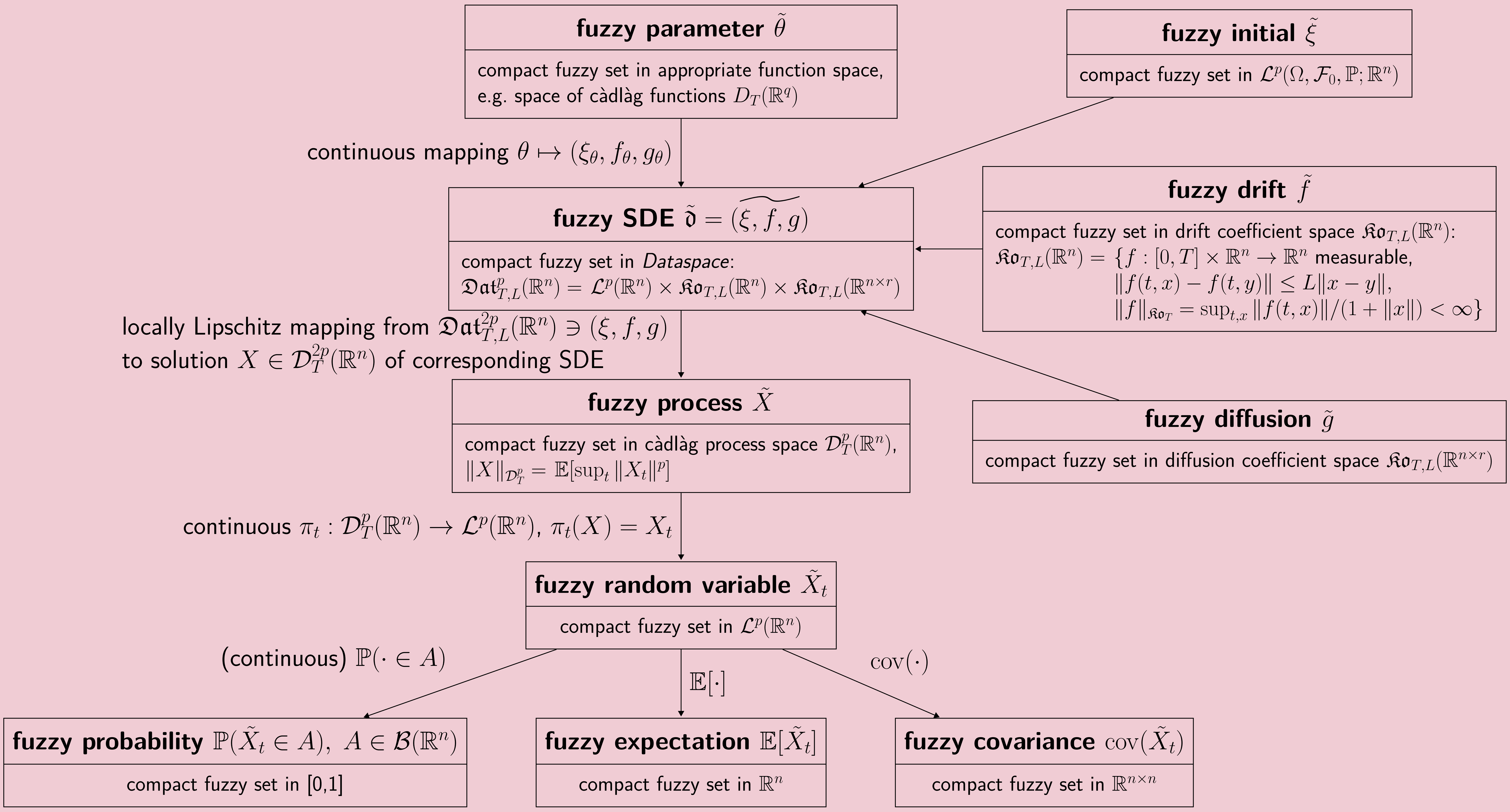
- Laws of natural systems are random and imprecise
  - a fox has a random chance to catch a rabbit (randomness)
  - the expected hunting performance of the foxes varies over time (fuzzyness)
- Fuzzy-valued SDEs can only handle crisp diffusion
  → fuzzyness of state is neglected in modelling randomness

New Approach

- Use fuzzy sets of mappings for
  - parameters, coefficients (data)
  - processes, random variables (solution)

rather than fuzzy-valued mappings
→ no technical problems, great modelling flexibility

General Scheme



fuzzy parameter $\tilde{\theta}$

compact fuzzy set in appropriate function space, e.g. space of càdlàg functions $D_T(\mathbb{R}^q)$

fuzzy initial $\tilde{\xi}$

compact fuzzy set in $\mathcal{L}^p(\Omega, \mathcal{F}_0, \mathbb{P}; \mathbb{R}^n)$

continuous mapping $\theta \mapsto (\xi_\theta, f_\theta, g_\theta)$

fuzzy SDE $\tilde{\mathfrak{d}} = \widetilde{(\xi, f, g)}$

compact fuzzy set in *Dataspace*:
$\mathfrak{Dat}^p_{T,L}(\mathbb{R}^n) = \mathcal{L}^p(\mathbb{R}^n) \times \mathfrak{Ko}_{T,L}(\mathbb{R}^n) \times \mathfrak{Ko}_{T,L}(\mathbb{R}^{n\times r})$

fuzzy drift $\tilde{f}$

compact fuzzy set in drift coefficient space $\mathfrak{Ko}_{T,L}(\mathbb{R}^n)$:
$\mathfrak{Ko}_{T,L}(\mathbb{R}^n) = \{f : [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$ measurable,
$\|f(t,x) - f(t,y)\| \le L\|x-y\|$,
$\|f\|_{\mathfrak{Ko}_T} = \sup_{t,x} \|f(t,x)\|/(1+\|x\|) < \infty\}$

locally Lipschitz mapping from $\mathfrak{Dat}^{2p}_{T,L}(\mathbb{R}^n) \ni (\xi, f, g)$
to solution $X \in \mathcal{D}^{2p}_T(\mathbb{R}^n)$ of corresponding SDE

fuzzy process $\tilde{X}$

compact fuzzy set in càdlàg process space $\mathcal{D}^p_T(\mathbb{R}^n)$,
$\|X\|_{\mathcal{D}^p_T} = \mathbb{E}[\sup_t \|X_t\|^p]$

fuzzy diffusion $\tilde{g}$

compact fuzzy set in diffusion coefficient space $\mathfrak{Ko}_{T,L}(\mathbb{R}^{n\times r})$

continuous $\pi_t : \mathcal{D}^p_T(\mathbb{R}^n) \to \mathcal{L}^p(\mathbb{R}^n)$, $\pi_t(X) = X_t$

fuzzy random variable $\tilde{X}_t$

compact fuzzy set in $\mathcal{L}^p(\mathbb{R}^n)$

(continuous) $\mathbb{P}(\cdot \in A)$

$\mathbb{E}[\cdot]$

$\operatorname{cov}(\cdot)$

fuzzy probability $\mathbb{P}(\tilde{X}_t \in A),\ A \in \mathcal{B}(\mathbb{R}^n)$

compact fuzzy set in [0,1]

fuzzy expectation $\mathbb{E}[\tilde{X}_t]$

compact fuzzy set in $\mathbb{R}^n$

fuzzy covariance $\operatorname{cov}(\tilde{X}_t)$

compact fuzzy set in $\mathbb{R}^{n\times n}$

Properties

- Beside uncertain range also uncertain behaviour and dependencies between drift and diffusion can be modelled
- Solutions of fuzzy SDEs can be understood as Lipschitz continuous mappings from $[0,T]$ to $\mathcal{F}_k(\mathcal{L}^p(\mathbb{R}^n))$
  → admits time-discrete approximations for e.g. fuzzy expectation $\mathbb{E}[\tilde{X}_t]$

Connections to other approaches

- Solutions of fuzzy SDEs can be interpreted as fuzzy-valued processes with a.s. continuous paths in special cases (additive noise, multiplicative noise, autonomous noise)
- Solutions of fuzzy SDEs are subsets of solution sets of corresponding fuzzy stochastic inclusions (α-cut-wise)

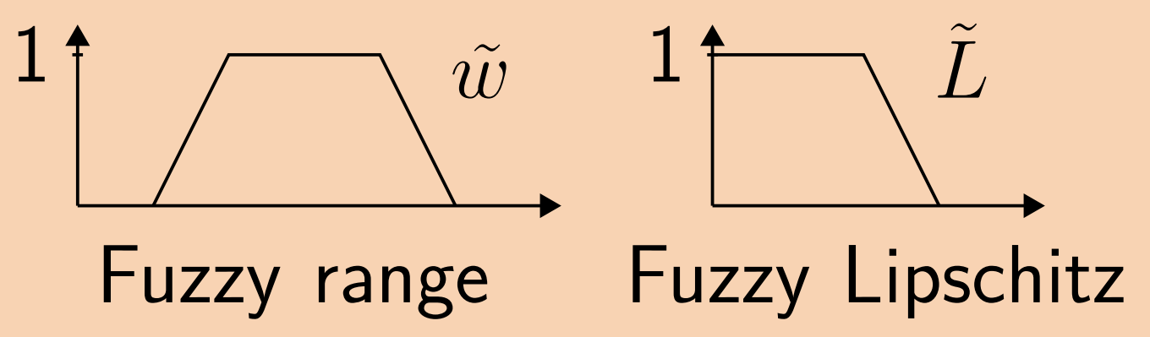
Example

In ecosystems parameters are usually not exactly measurable and vary over time, too. Therefore we consider a stochastic predator-prey system with uncertain parameters

$$dX_t = \left(r\,\frac{k - X_t}{k} - a\,Y_t\right) X_t\,dt + \sigma_1 X_t\,dB^{(1)}_t,\ X_0 = x_0 \in \mathbb{R},$$

$$dY_t = \left(-m + b_t\,X_t\,\frac{Y_t}{z + Y_t}\right) Y_t\,dt + \sigma_2 Y_t\,dB^{(2)}_t,\ Y_0 = y_0 \in \mathbb{R}$$

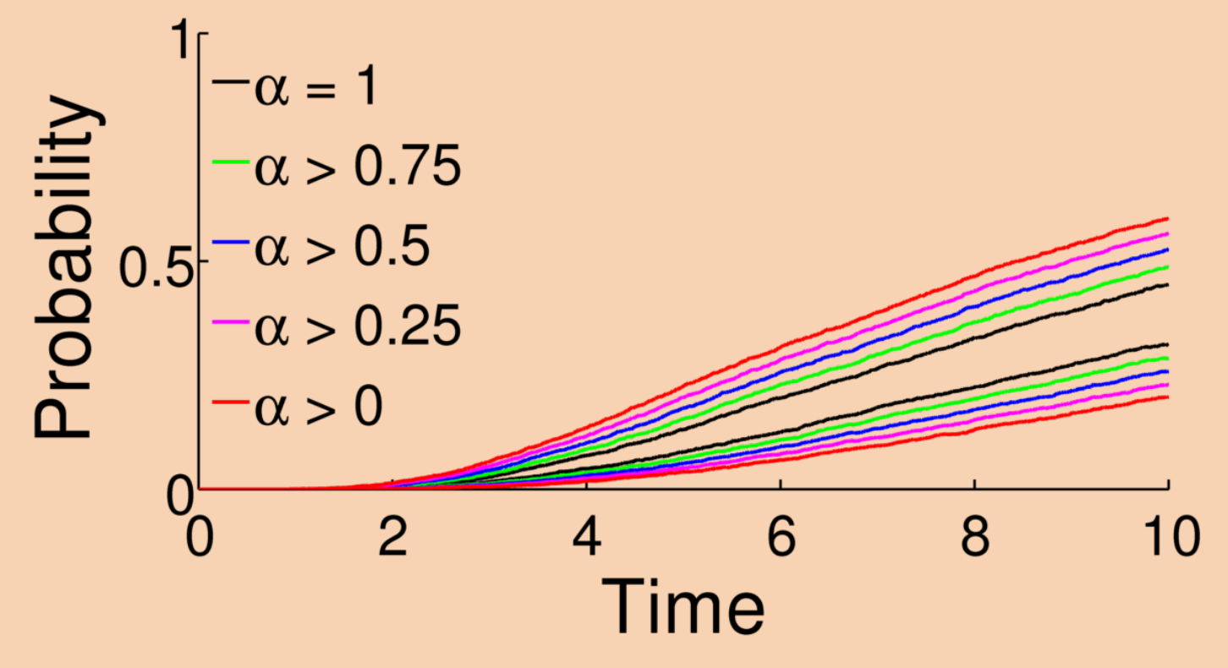
Modelling $\tilde{\sigma}_2$ as fuzzy number and $\tilde{b}$ as a fuzzy set of continuous functions:



Fuzzy range Fuzzy Lipschitz

$$\tilde{b}(f) = \min\left(\inf_t\ \tilde{w}(f_t)\,,\ \inf_{s\neq t} \tilde{L}\left(\tfrac{|f_t - f_s|}{|t-s|}\right)\right)$$

Of interest is the probability of extinction, consider $\mathbb{P}(\tilde{Y}_t < y_{min})$:



[*] TU Bergakademie Freiberg
Fakultät für Mathematik und Informatik
Institut für Numerische Mathematik und Optimierung
D-09596 Freiberg

[**] TU Bergakademie Freiberg
Fakultät für Mathematik und Informatik
Institut für Stochastik
D-09596 Freiberg