Rationalizability under Uncertainty

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Overview

- Nash equilibrium requires that each player has a correct belief about the other players' strategy choices and behaves optimally given her belief. This is too restrictive in the sense that it rules out behaviors that do not contradict the rationality of the players.
- Bernheim (1984) and Pearce (1984) independently propose the solution concept called rationalizability as an appropriate criterion for choosing rational behavior in noncooperative strategic situations.
- Here I generalize the concept of rationalizability by using sets of probabilities to model uncertainty in games, and examine how game theory can be informed by incorporating imprecise probability.

Game-Theoretic Preliminaries

- A finite normal form game $G = \langle I, \{S_i\}, \{u_i\} \rangle_{i \in I}$ consists of:
  - I: a finite set of players who make decisions
  - S_i: a finite set of actions of player i (pure strategies)
  - $u_i : S \to \mathbb{R}$ denotes player i's payoff function, where $S = \prod_{i \in I} S_i$.
- Let Δ_i denote the set of player i's mixed strategies, which can be regarded as probability measures on S_i.

Basic Notions

- Basically, the solution concept rationalizability captures the idea of rational behavior constrained only by the common knowledge that each player maximizes expected utility with respect to a single personal probability distribution representing uncertainty.
- A belief of player i about the other players' strategy choices is a probability distribution over the set of actions $S_{-i} = \prod_{j \neq i} S_j$.
- A strategy $\delta_i \in \Delta_i$ is said to be rational if there exists a belief $\delta_{-i} \in \Delta_{-i}$ such that δ_i maximizes player i's expected payoff. In this case, δ_i is called a best response to the belief δ_{-i}.

Definition: Rationalizability

In a game G, an action s_i of player i is rationalizable if for each player $j \in I$ there exists a set Z_j of actions such that

1. $s_i \in Z_i$,
2. For each player $j \in I$, every action s_j in Z_j is a *best response* to a belief of player j that assigns positive probability only to those actions in Z_{-j}.

Properties of Rationalizable Actions

- **Proposition**: Every action used with positive probability in some mixed strategy Nash equilibrium is rationalizable.
- It immediately follows from the existence of Nash equilibrium for finite games that rationalizable actions always exists.
- **Corollary**: In finite games, each player i's set of rationalizable actions is nonempty.

Algorithm for Rationalizability

Proposition: The set of strategy profiles survives iterated elimination of strictly dominated actions is equal to the set of profiles of rationalizable actions.

- Note that row player's action M is never a best response to any precise conjecture over $\{L, R\}$.
- Thus, the only rationalizable actions for both players are D and R respectively.

	L	R
U	$10, 1$	$0, 2$
M	$4, 10$	$4, 1$
D	$0, 1$	$10, 2$

Rationalizability under Uncertainty

- In analogy with rationalizability, the new solution concept we call Γ-maximin rationalizability captures the idea that each player believes that her opponents maximize their own minimun expected payoff with respect to their conjectures about the other players' strategy choices.
- A conjecture C_{-i} of player i about her opponents' strategy choices is represented by a nonempty, closed, and convex set of probability measures on S_{-i}.
- A strategy $\delta_i \in \Delta_i$ is called rational under uncertainty if there exists a conjecture C_{-i} such that δ_i maximizes player i's minimum expected payoff with respect to C_{-i}. In this case, we say that δ_i is Γ-maximin admissible relative to C_{-i}.

Definition: Γ-maximin Rationalizability

In a game G, an action s_i of player i is Γ-maximin rationalizable if for each player $j \in I$ there exists a set A_j of actions such that

1. $s_i \in A_i$,
2. For each player $j \in I$, every action s_j in A_j is Γ-*maximin admissible* relative to a conjecture C_{-j} of player j such that each probability measure in C_{-j} assigns positive probability only to those actions in A_{-j}.

Properties of Γ-maximin Rationalizability

- **Proposition**: In finite games, each player i's set of Γ-maximin rationalizable actions is nonempty.
- The decision rule Γ-maximin has maximization of expected payoff as a special case when probability is determinate, i.e., when $\mathcal{P}$ contains a single probability distribution.
- **Corollary**: For each player i, if player i's action δ_i is rationalizable, then it is Γ-maximin rationalizable.

Example: Difference from Rationalizability

Consider again the game mentioned above. Now assume that each player's feasible options are pure strategies only, that is, explicit randomization is excluded.

- Recall that only D and R are rationalizable.
- However, all actions of both player are Γ-maximin rationalizable, especially the action M.

	L	R
U	$10, 1$	$0, 2$
M	$4, 10$	$4, 1$
D	$0, 1$	$10, 2$

- Let us verify how the action M can be Γ-maximin rationalized in the sense defined above.
  - Consider the sets of actions: $A_1 = \{U, M\}$ and $A_2 = \{L, R\}$.
  - Let the corresponding conjecture sets: $C_1 = \{\mathbb{P}_1(\cdot) : 0 \leq \mathbb{P}_1(R) \leq \frac{3}{5}\}$ and $C_2 = \{\mathbb{P}_2(\cdot) : \mathbb{P}_2(D) = 0, 0 \leq \mathbb{P}_2(U) \leq 1\}$.
  - Both U and M are Γ-maximin admissible relative to C_1; and both L and R are Γ-maximin admissible relative to C_2.

Properties Continued

- When is the notion of rationalizability equivalent to the concept of Γ-maximin rationalizability?
- **Proposition**: If each player's choice set is convex, then the set of rationalizable actions is equal to the set of Γ-maximin rationalizable.
- It has been shown (Walley, 1990) that when the choice set is convex and the convex set of probabilities $\mathcal{P}$ is closed, then Γ-maximin admissible options are Bayes-admissible.