Accept & reject statement-based uncertainty models

Generalizing nonstrict & strict preference desirability

1 Context

Agent faced with uncertainty;

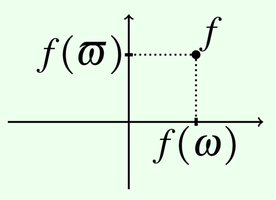
Possibility space Ω,

e.g., set of experimental outcomes;

Gambles real-valued functions f on Ω;

interest in a linear space of gambles $\mathcal{L}$.

Example format used for illustrations: $\Omega := \{\omega, \varpi\}$, and $\mathcal{L}$ is the linear space of all gambles on Ω.



Gamble set operations & hulls

Let $f \in \mathcal{L}$ and $\mathcal{K}, \mathcal{K}' \subseteq \mathcal{L}$, then:

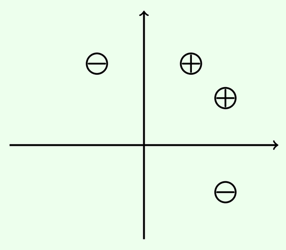
- negation $-\mathcal{K} := \{-g : g \in \mathcal{K}\}$,
- ray $\bar{f} := \{\lambda f : \lambda \in \mathbb{R}_{>0}\}$,
- positive scalar hull $\overline{\mathcal{K}} := \bigcup_{f\in\mathcal{K}} \bar{f}$,
- Minkowski addition: $\mathcal{K} + \mathcal{K}' := \{g + h : g \in \mathcal{K}, h \in \mathcal{K}'\}$,
- positive linear hull $\operatorname{posi}\mathcal{K} := \bigcup\{\sum_{g\in\mathcal{K}''} \bar{g} : \mathcal{K}'' \subseteq \mathcal{K}, |\mathcal{K}''| \in \mathbb{N}\}$

2 Accepting & rejecting

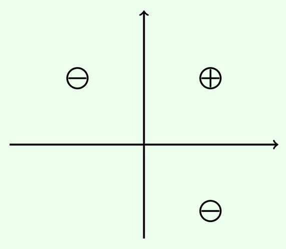
The agent gives an assessment by making statements about gambles f:

Accepting ($\oplus$) implies a commitment:

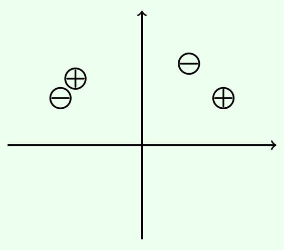
(i) outcome $\omega \in \Omega$ is determined,
(ii) he receives the payoff $f(\omega)$.



Rejecting ($\ominus$) implies that he considers accepting f unreasonable; this is relevant when combining assessments.



An assessment is a pair $\mathcal{A} := \langle \mathcal{A}_{\geq}; \mathcal{A}_{<} \rangle$ in $\mathbf{A} := 2^{\mathcal{L}} \times 2^{\mathcal{L}}$ of sets of acceptable respectively dispreferred gambles.



Unresolved gambles belong to neither category: $\mathcal{A}_{\smile} := (\mathcal{A}_{\geq} \cup \mathcal{A}_{<})^{c}$;

Confusing gambles belong to both categories: $\mathcal{A}_{\natural} := \mathcal{A}_{\geq} \cap \mathcal{A}_{<}$.

No confusion is present in an assessment $\mathcal{A}$ if and only if $\mathcal{A}_{\natural} = \varnothing$.

The set of assessments without confusion is $\mathbb{A}$.

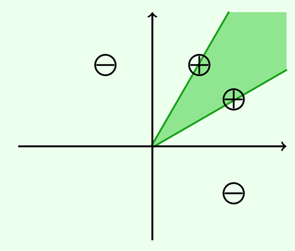
Indifferent gambles are acceptable and have an acceptable negation: $\mathcal{A}_{\simeq} := -\mathcal{A}_{\geq} \cap \mathcal{A}_{\geq}$;

Favorable gambles are acceptable and have a dispreferred negation: $\mathcal{A}_{>} := -\mathcal{A}_{<} \cap \mathcal{A}_{\geq}$;

Incomparable gambles are unresolved and have an unresolved negation: $\mathcal{A}_{\asymp} := -\mathcal{A}_{\smile} \cap \mathcal{A}_{\smile}$.

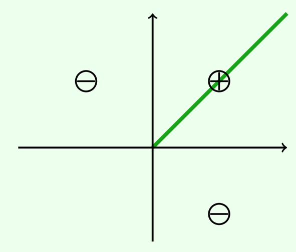
3 Deductive closure

We assume gamble payoffs are expressed in a linear precise utility. This implies:



Positive scaling If f is acceptable, then all gambles in the ray $\bar{f}$ are acceptable;

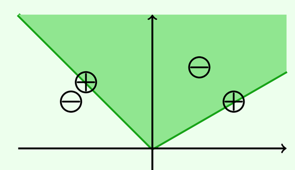
Combination If f and g are acceptable, then $f + g$ is acceptable.



So we can extend an assessment $\mathcal{A}$:

Deductive extension

$$\operatorname{ext}_{\mathbf{D}}\mathcal{A} := \langle \operatorname{posi}\mathcal{A}_{\geq}; \mathcal{A}_{<} \rangle;$$



extension legend:

acceptable (hull) gambles; included (border) ray

Deductive closure of an assessment $\mathcal{D}$ if and only if $\operatorname{ext}_{\mathbf{D}}\mathcal{D} = \mathcal{D}$ or $\operatorname{posi}\mathcal{D}_{\geq} = \mathcal{D}_{\geq}$.

The deductively closed assessments form the set $\mathbf{D} \subset \mathbf{A}$; those without confusion form the set $\mathbb{D} := \mathbf{D} \cap \mathbb{A}$.

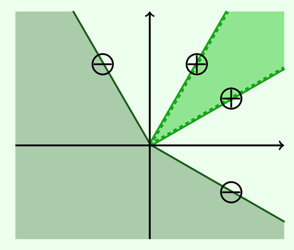
Deductively closable assessments are those that remain without confusion; they form the set $\mathbb{A}^{\!+} := \{\mathcal{A} \in \mathbb{A} : \operatorname{ext}_{\mathbf{D}}\mathcal{A} \in \mathbb{D}\}$.

Removing confusion from an assessment $\mathcal{D}$ in $\mathbf{D}$ can be done automatically: both $\operatorname{ext}_{\mathbf{D}}\langle \mathcal{D}_{\geq} \setminus \mathcal{D}_{\natural}; \mathcal{D}_{<} \setminus \mathcal{D}_{\natural} \rangle$ and $\langle \mathcal{D}_{\geq}; \mathcal{D}_{<} \setminus \mathcal{D}_{\natural} \rangle$ belong to $\mathbb{D}$.

4 No limbo

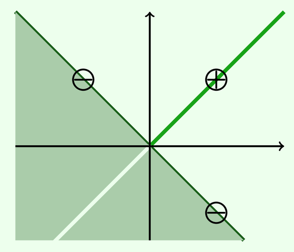
Let $\mathcal{D}$ in $\mathbf{D}$ and f in $\mathcal{D}_{\smile}$, then without increasing confusion f can

- always be rejected,
- can be accepted if and only if $f \notin \operatorname{ina}\mathcal{D} := (\overline{\mathcal{D}_{<} \setminus \mathcal{D}_{\geq}}) - (\mathcal{D}_{\geq} \cup \{0\})$.



Limbo is the set of unresolved gambles

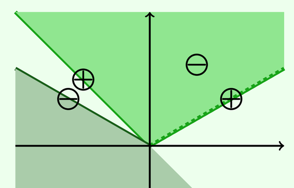
$$(\operatorname{ina}\mathcal{D}) \setminus \mathcal{D}_{<}$$



that cannot additionally be acceptable without increasing confusion under deductive closure.

Reckoning extension

$$\operatorname{ext}_{\mathbf{M}}\mathcal{D} := \langle \mathcal{D}_{\geq}; \mathcal{D}_{<} \cup \operatorname{ina}\mathcal{D} \rangle;$$



extension legend:

favorable (hull) gambles; included favorable (border) ray; dispreferred (hull) gambles; included dispreferred (border) ray

No limbo is present in a deductively closed assessment $\mathcal{M}$ if and only if $\operatorname{ext}_{\mathbf{M}}\mathcal{M} = \mathcal{M}$ or $\operatorname{ina}\mathcal{M} \subseteq \mathcal{M}_{<}$.

Models $\mathcal{M}$ are deductively closed assessments without limbo.

Models form the set $\mathbf{M} \subset \mathbf{D}$; those without confusion form the set $\mathbb{M} := \mathbf{M} \cap \mathbb{A}$.

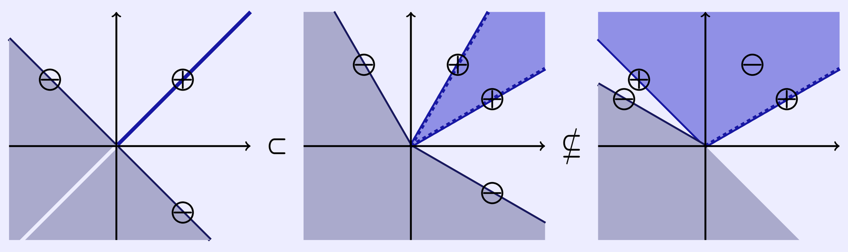
5 Model properties

Given a model $\mathcal{M}$ in $\mathbb{M}$, then

- $\operatorname{ina}\mathcal{M} = \mathcal{M}_{<} - (\mathcal{M}_{\geq} \cup \{0\})$,
- $\overline{\mathcal{M}_{<}} = \mathcal{M}_{<}$ and $\operatorname{posi}\mathcal{M}_{>} = \mathcal{M}_{>}$,
- $\mathcal{M}_{>} + \mathcal{M}_{\geq} = \mathcal{M}_{>}$.

6 Set relations & operations

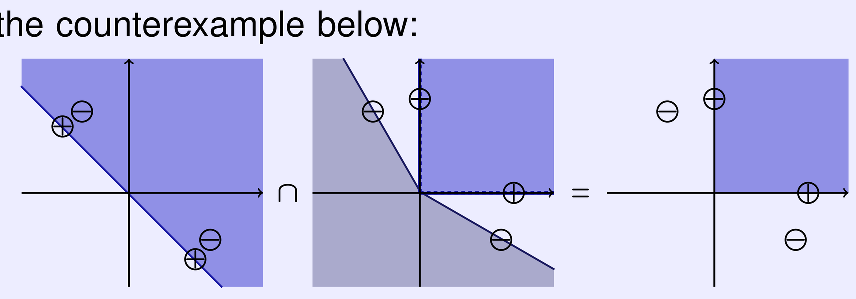
'Less committal than'-relation $\subset$ is an assessment relation derived from inclusion: $\mathcal{A} \subset \mathcal{B} \Leftrightarrow \mathcal{A} \subseteq \mathcal{B} \wedge \mathcal{A} \neq \mathcal{B}$ and $\mathcal{A} \subseteq \mathcal{B} \Leftrightarrow \mathcal{A}_{\geq} \subseteq \mathcal{B}_{\geq} \wedge \mathcal{A}_{<} \subseteq \mathcal{B}_{<}$.



Assessment union $\bigcup$ is componentwise set union;

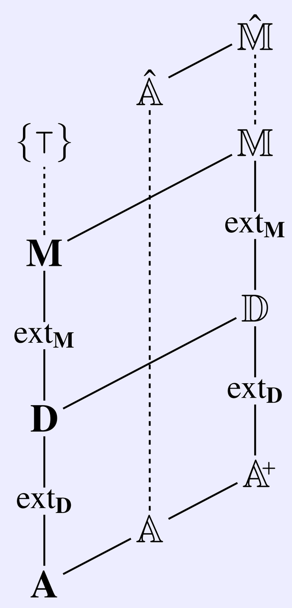
Assessment intersection $\bigcap$ is componentwise set intersection;

The sets $\mathbb{A}$, $\mathbb{A}^{\!+}$, $\mathbf{D}$, $\mathbb{D}$, and $\mathbb{M}$ are closed under arbitrary non-empty intersections, but $\mathbf{M}$ is not, as is attested by the counterexample below:



7 Order-theoretic results

The 'less committal than or equal'-relation $\subseteq$ engenders a partial ordering of the assessments.



Dominating assessments in a set of assessments $\mathbf{B} \subseteq \mathbf{A}$:

$$\mathbf{B}_{\mathcal{A}} := \{\mathcal{B} \in \mathbf{B} : \mathcal{A} \subseteq \mathcal{B}\}.$$

($\mathbf{B}_{\perp} = \mathbf{B}$ with $\perp := \langle \varnothing; \varnothing \rangle$)

Maximal assessments $\hat{\mathbf{B}}$ are the undominated ones in $\mathbf{B}$;

- $\hat{\mathbf{A}} = \hat{\mathbf{D}} = \hat{\mathbf{M}} = \{\top\}$ with $\top := \langle \mathcal{L}; \mathcal{L} \rangle$,
- $\hat{\mathbb{A}} = \{\langle \mathcal{K}; \mathcal{L} \setminus \mathcal{K} \rangle : \mathcal{K} \subseteq \mathcal{L}\}$, and
- $\hat{\mathbb{M}} = \hat{\mathbb{D}} = \hat{\mathbb{A}}^{\!+} = \hat{\mathbb{A}} \cap \mathbb{A}^{\!+}$.

Intersection structures is what the posets $(\mathbf{A}, \subseteq)$, $(\mathbb{A}, \subseteq)$, $(\mathbb{A}^{\!+}, \subseteq)$, $(\mathbf{D}, \subseteq)$, $(\mathbb{D}, \subseteq)$, and $(\mathbb{M}, \subseteq)$ are, because the sets are closed under intersection.

Closure operators are associated to each intersection structure $(\mathbf{B}, \subseteq)$ by $\operatorname{cl}_{\mathbf{B}}\mathcal{A} := \bigcap \mathbf{B}_{\mathcal{A}}$ with $\bigcap\varnothing = \top$;

- $\operatorname{cl}_{\mathbf{B}} = \operatorname{id}$ on $\mathbf{B} \cup \{\top\}$,
- $\operatorname{cl}_{\mathbf{D}} = \operatorname{ext}_{\mathbf{D}}$ and $\operatorname{cl}_{\mathbb{A}}$ returns $\top$ outside of $\mathbb{A}$,
- $\operatorname{cl}_{\mathbb{A}^{\!+}} = \operatorname{id}$, $\operatorname{cl}_{\mathbb{D}} = \operatorname{ext}_{\mathbf{D}}$, and $\operatorname{cl}_{\mathbb{M}} = \operatorname{ext}_{\mathbf{M}} \circ \operatorname{ext}_{\mathbf{D}}$ on $\mathbb{A}^{\!+}$ and it returns $\top$ elsewhere.

Complete lattices is what the posets $(\mathbf{A}, \subseteq)$, $(\mathbb{A} \cup \{\top\}, \subseteq)$, $(\mathbb{A}^{\!+} \cup \{\top\}, \subseteq)$, $(\mathbf{D}, \subseteq)$, $(\mathbb{D} \cup \{\top\}, \subseteq)$, and $(\mathbb{M} \cup \{\top\}, \subseteq)$ become with $\bigcap$ as the infimum operator and $\operatorname{cl}_{\mathbf{B}} \circ \bigcup$ as the supremum operator;

- deductive union $\uplus := \operatorname{cl}_{\mathbf{D}} \circ \bigcup$,
- reckoning union $⊞ := \operatorname{cl}_{\mathbf{M}} \circ \bigcup$.

8 Dominating models

The agent specifies an assessment $\mathcal{A}$; if it is an element of $\mathbb{A}^{\!+}$, then it can be extended to a model without confusion: $\mathcal{M} := \operatorname{cl}_{\mathbb{M}}\mathcal{A} = \operatorname{ext}_{\mathbf{M}}(\operatorname{ext}_{\mathbf{D}}\mathcal{A}) \in \mathbb{M}$. Therefore knowing whether $\mathcal{A}$ is in $\mathbb{A}^{\!+}$ is important.

Characterization of $\mathbb{A}^{\!+}$ using the set of dominating maximal elements $\hat{\mathbb{M}}_{\mathcal{A}}$:

$$\mathcal{A} \in \mathbb{A}^{\!+} \Leftrightarrow \hat{\mathbb{M}}_{\mathcal{A}} \neq \varnothing.$$

This means that if all assessments in some family are dominated by a common model in $\mathbb{M}$, then their reckoning union is a model in $\mathbb{M}$.

Inferences can be drawn from any assessment $\mathcal{A}$ as well using $\hat{\mathbb{M}}_{\mathcal{A}}$:

$$\operatorname{cl}_{\mathbb{M}}\mathcal{A} = \bigcap \hat{\mathbb{M}}_{\mathcal{A}}.$$

Our models in $\mathcal{M}$ are compatible with AGM-style belief change and revision.

9 Smallest models

Some a priori assumptions can be captured by positing a smallest model $\mathcal{S}$ in $\mathbb{M}$ that replaces $\perp$: so we work in $\mathbb{M}_{\mathcal{S}}$ instead of $\mathbb{M}$.

(All the results above remain valid, mutatis mutandis.)

Natural extension of an assessment $\mathcal{A}$ is its reckoning union with the smallest model:

$$\mathcal{A} ⊞ \mathcal{S}.$$

Coherent models $\mathcal{M}$ coincide with their natural extension: $\mathcal{M} ⊞ \mathcal{S} = \mathcal{M}$ or $\mathcal{S} \subseteq \mathcal{M}$.

10 Universal a priori assumption

Given the commitments implied by accepting gambles, there is one assumption we judge reasonable to posit.

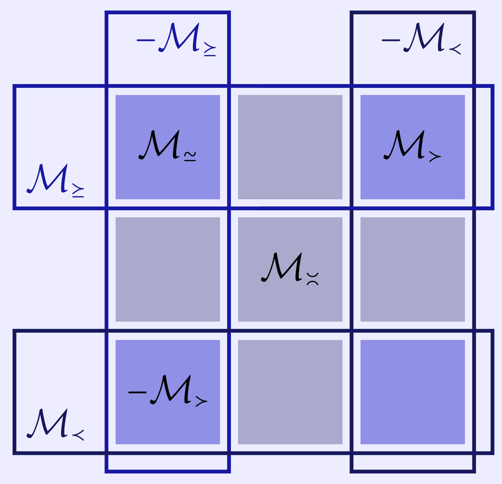
Indifference to status quo corresponds to the zero gamble 0 being acceptable, and therefore indifferent:

$$\mathcal{O} \subseteq \mathcal{S},$$

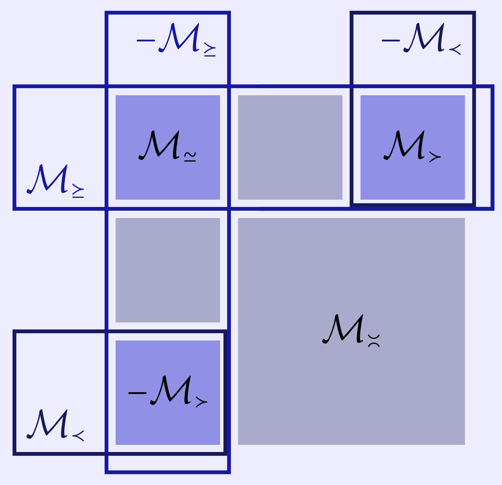
where $\mathcal{O} := \langle \{0\}; \varnothing \rangle$.

11 Partitions

All models $\mathcal{M}$ in $\mathbb{M}$ partition gamble space $\mathcal{L}$ into nine elements—some possibly empty—depending on whether a gamble and its negation are acceptable, dispreferred, or neither.



The model is maximal if and only if the more grayish partition elements are empty.



The framework can be simplified by replacing rejection by favorability: $-\mathcal{M}_{<} \subseteq \mathcal{M}_{\geq}$.

12 Basic gamble relations

Fix a model $\mathcal{M}$ in $\mathbb{M}_{\mathcal{O}}$, then we can define the following relations between gambles f and g in $\mathcal{L}$:

f* is accepted in exchange for *g

$$f \geq g \Leftrightarrow f - g \geq 0 \Leftrightarrow f - g \in \mathcal{M}_{\geq};$$

f* is dispreferred to *g

$$f < g \Leftrightarrow f - g < 0 \Leftrightarrow f - g \in \mathcal{M}_{<}.$$

The 'axioms' defining $\mathbb{M}_{\mathcal{O}}$—no confusion, deductive closure, no limbo, and indifference to status quo—can then be reformulated:

No confusion $f \ngeq g \vee f \nless g$,

Reflexivity $f \geq f$,

Transitivity $f \geq g \wedge g \geq h \Rightarrow f \geq g$, $f < g \wedge h \geq g \Rightarrow f < h$,

Mixture independence for all $\square \in \{\geq, <\}$ and $0 < \mu \leq 1$:

$$f \,\square\, g \Leftrightarrow \mu f + (1-\mu)h \,\square\, \mu g + (1-\mu)h.$$

So we can conclude:

- acceptability $\geq$ is reflexive and transitive, and thus a non-strict preorder and a vector ordering,
- dispreference $<$ is irreflexive.

13 Derived gamble relations

A number of other useful gamble relations follow from the basic ones:

Indifference between *f* and *g*

$$f \simeq g \Leftrightarrow f \geq g \wedge g \geq f \Leftrightarrow f - g \in \mathcal{M}_{\simeq};$$

f* is preferred to *g

$$f > g \Leftrightarrow f \geq g \wedge g < f \Leftrightarrow f - g \in \mathcal{A}_{>};$$

***f* and *g* are incomparable**

$$f \asymp g \Leftrightarrow f - g \in \mathcal{A}_{\asymp}.$$

We can then conclude using the 'axioms' regulating the basic relations $\geq$ and $<$—no confusion, reflexivity, transitivity, and mixture independence—that:

- indifference $\simeq$ is reflexive, transitive, and symmetric, and thus an equivalence relation,
- preference $>$ is irreflexive and transitive, and thus a strict partial order ideally suited for decision making,
- incomparability $\asymp$ is irreflexive, symmetric, and in general intransitive.

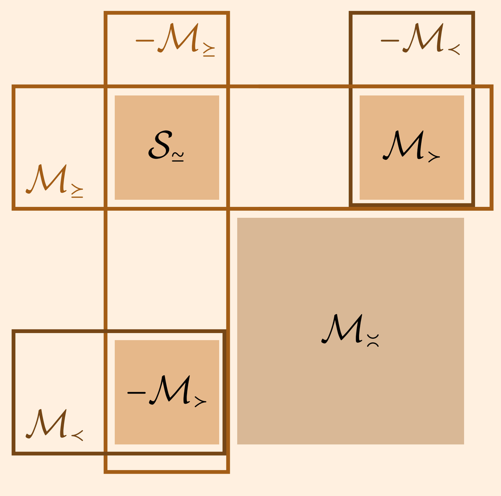
14 Smallest models

Associate the relations $\trianglerighteq$ and $\triangleleft$ with the smallest model $\mathcal{S}$ in $\mathbb{M}_{\mathcal{O}}$, then a necessary condition for coherence is

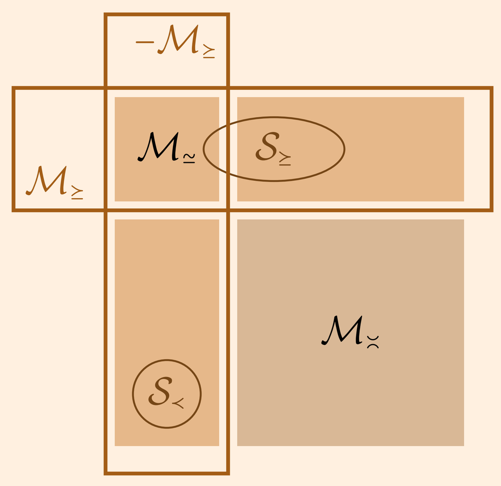
Monotonicity $f \trianglerighteq g \Rightarrow f \geq g$ and $f \triangleleft g \Rightarrow f < g$.

15 Desirability

Strict preference desirability is a simplification with $\mathcal{M}$ in $\mathbb{M}_{\mathcal{O}}$ such that $\mathcal{M}_{\geq} = \mathcal{M}_{>} \cup \mathcal{S}_{\simeq}$.



Nonstrict preference desirability with $\mathcal{M} = \operatorname{cl}_{\mathbb{M}}\langle \mathcal{M}_{\geq} \cup \mathcal{S}_{\geq}; \mathcal{S}_{<} \rangle$.



16 Comparison with the literature

Forms of what we call strict preference desirability have gotten the most attention in the literature:

Smith (1961, §14) talks about 'exchange vectors' (finite Ω), works with $\mathcal{S}_{>} := \mathcal{L}_{>0}$, and imposes that $\mathcal{M}_{>}$ is open;

Seidenfeld et al. (1990, §IV) talk about 'favorable' gambles (finite Ω) and work with $\mathcal{S}_{>} := \mathcal{L}_{>0}$;

Walley (1991, §3.7.8) discusses 'strictly desirable' gambles, works with $\mathcal{S}_{>} := \mathcal{L}_{>0}$, and imposes an openness axiom $\mathcal{M}_{>} \setminus \mathcal{S}_{>} \subseteq \mathcal{M}_{>} + \mathbb{R}_{>0}$;

Walley (2000, §6) drops the openness axiom and advocates a desirability framework with $\mathcal{S}_{>} := \mathcal{L}_{>0}$;

De Cooman & Quaeghebeur (2009–2011) build on this, but are the first with a nontrivial $\mathcal{S}_{\simeq}$, i.e., the gambles expressing exchangeability.

Occurrences of the nonstrict case are also important:

Williams (1974) talks about 'acceptable bets' and works with an $\mathcal{S}$ defined by $\mathcal{S}_{>} := \{f \in \mathcal{L} : \inf f > 0\}$, so there is no default indifference to status quo;

Walley (1991, §3.7.3) discusses 'almost desirable' gambles, works with $\mathcal{S} := \langle \mathcal{L}_{\geq 0}; \{f \in \mathcal{L} : \sup f < 0\} \rangle$, and imposes a closure axiom $f + \mathbb{R}_{>0} \subseteq \mathcal{M}_{\geq} \Rightarrow f \in \mathcal{M}_{\geq}$;

Walley (1991, App. F) talks about 'really desirable' gambles and works with $\mathcal{S} := \langle \mathcal{L}_{\geq 0}; \{f \in \mathcal{L} : \sup f < 0\} \rangle$.

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