Rationalizability under Uncertainty

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Abstract

The game-theoretic solution concept called *rationalizability* ([1], [4]) captures the idea of rational behavior constrained only by the common knowledge that each player maximizes expected utility with respect to a single personal probability distribution representing uncertainty. Here I generalize the concept of rationalizability by using sets of probabilities to model uncertainty in games, and examine how game theory can be informed by introducing imprecise probability when it is common knowledge among players that each player maximizes the minimum expectation (known as Γ-*maximin*, see [2]).

Consider a finite normal form game $G = \langle I, \{S_i\}, \{u_i\}\rangle_{i\in I}$, where I denotes a finite set of players, S_i denotes the finite set of actions of player i, and $u_i : S \to \mathbb{R}$ denotes player i's payoff function (where $S = \prod_{i\in I} S_i$). And let Δ_i denote the set of player i's mixed strategies, which can be regarded as probability measures on S_i.

Rationalizability requires that each player maximizes her own expected payoff against her belief about the opponents' strategy choices. A *belief* of player i about the other players' strategy choices in a game G is a probability distribution over the set of actions $S_{-i} = \prod_{j\neq i} S_j$. Note that this formulation of beliefs allows a player to hold a belief that the other players' actions are *correlated*. We say that a strategy $\delta_i \in \Delta_i$ is *rational* if there exists a belief $\delta_{-i} \in \Delta_{-i}$ $(= \prod_{j\neq i} \Delta_j)$ such that δ_i maximizes player i's expected payoff. In this case, δ_i is called a *best response* to the belief δ_{-i}. We then formulate the concept of rationalizability as follows ([3]).

Definition 1 (Rationalizability) *In a game G, an action s_i of player i is rationalizable if for each player $j \in I$ there exists a set Z_j of actions such that (i) $s_i \in Z_i$, and (ii) for each player $j \in I$, every action s_j in Z_j is a best response to a belief δ_{-j} of player j that assigns positive probability only to those actions in Z_{-j}.*

In analogy with rationalizability, the new solution concept we call Γ-*maximin rationalizability* captures the idea that each player believes that her opponents maximizes their own minimun expected payoff with respect to their conjectures about the other players' strategies. A *conjecture* C_{-i} of player i about her opponents' strategy choices is a nonempty, closed, and convex set of probability measures on S_{-i}. And a strategy $\delta_i \in \Delta_i$ is called *rational under uncertainty* if there exists a conjecture C_{-i} such that δ_i maximizes player i's minimum expected payoff with respect to C_{-i}. In this case, we say that δ_i is Γ-*maximin admissible* relative to C_{-i}. We then define:

Definition 2 (Γ-maximin Rationalizability) *In a game G, an action s_i of player i is Γ-maximin rationalizable if for each player $j \in I$ there exists a set A_j of actions such that (i) $s_i \in A_i$, and (ii) for each player $j \in I$, every action s_j in A_j is Γ-maximin admissible relative to a conjecture C_{-j} of player j such that each probability measure in C_{-j} assigns positive probability only to those actions in A_{-j}.*

Clearly, Γ-maximin rationalizabilty has rationalizability as a special case when all players' conjectures are comprised by a single probability measure. In order to illustrate the difference between these two solution concepts, consider the 3 × 2 game shown to the left. Assume that each player's feasible options are pure strategies only, that is, explicit randomization is excluded; no non-trivial mixed strategy is feasible for each player.

	L	R
U	$10, 1$	$0, 2$
M	$4, 10$	$4, 1$
D	$0, 1$	$10, 2$

Note that row player's action M is never a best response to any *precise* conjecture over $\{L, R\}$. Thus, the only rationalizable actions for both players are D and R respectively. However, all actions in this game are Γ-maximin rationalizable. The crucial part of this claim is to argue that row player's action M is Γ-maximin rationalizable. This can be shown by considering the following case: let $A_1 = \{U, M\}$ and $A_2 = \{L, R\}$ be the sets of actions for row and column player respectively. Assume that row and column player's conjecture is depicted respectively by $C_{-1} = \{\mathbb{P}_1(\cdot) : 0 \le \mathbb{P}_1(R) \le 0.6\}$ and $C_{-2} = \{\mathbb{P}_2(\cdot) : \mathbb{P}_2(D) = 0, 0 \le \mathbb{P}_2(U) \le 1\}$. Then, the action M is Γ-maximin rationalizable. Based on this, it is easy to verify that all the other actions of both players are Γ-maximin rationalizable.

Keywords. Uncertainty, sets of probabilities, game theory, rationalizability, Γ-maximin rationalizability.

References

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